The Geometry of 2×2 Matrices

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Spring 2009 MD-DC-VA Sectional Meeting
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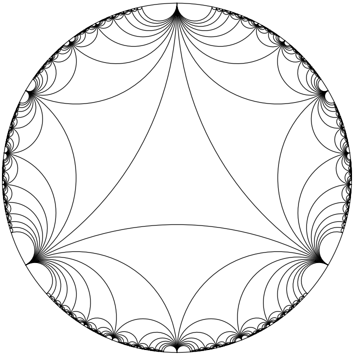

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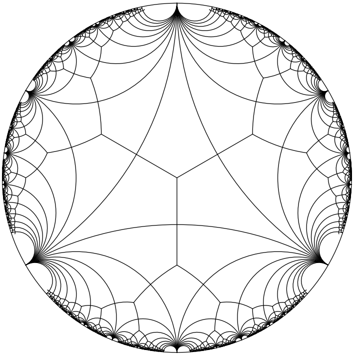

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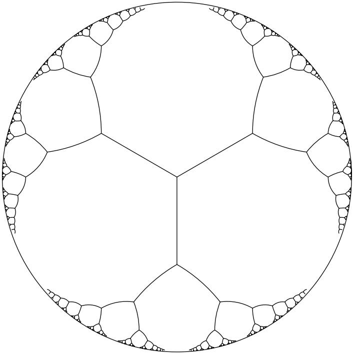

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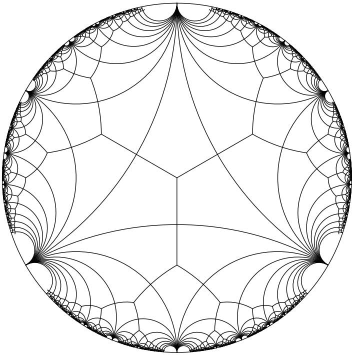

Algebraicizing geometry through symmetry

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Felix Klein's *Erlangen Program:* A *geometry* is the study of objects invariant under some group of *symmetries.* (1872)

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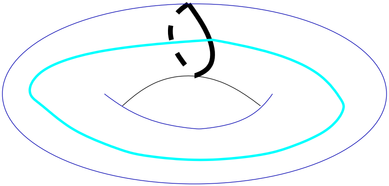

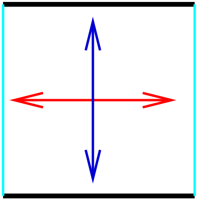

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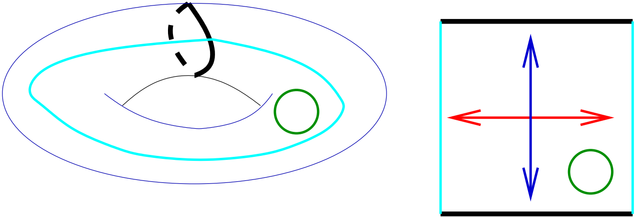

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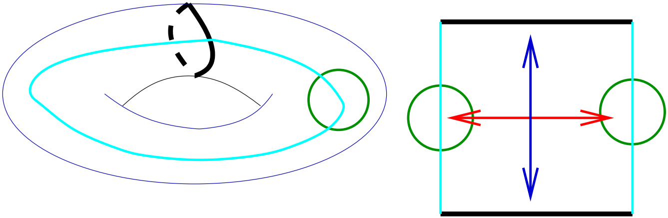

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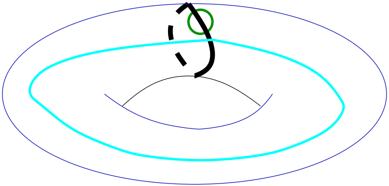

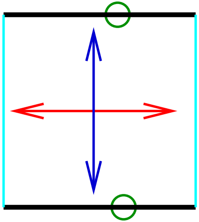

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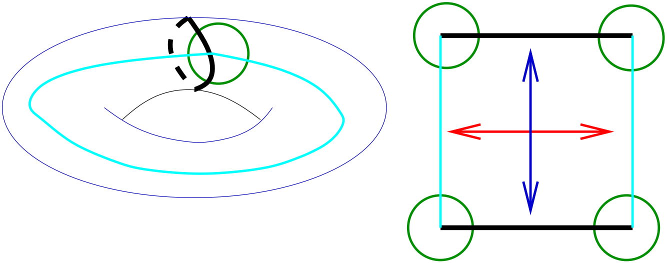

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- For example a cube has the topology of a sphere, but its geometry fails to be Euclidean at its 8 vertices.

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- In general the space of equivalence classes of a geometry has an interesting geometry of its own.

Euclidean structures on torus

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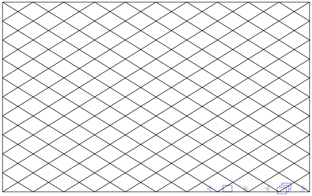

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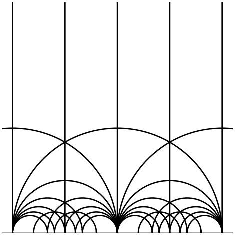
- Changing basis $\longleftrightarrow$ action of the group $\mathrm{SL}(2,\mathbb{Z})$ of integral 2x2 matrices by

$$\tau \longmapsto \frac{a\tau + b}{c\tau + d} \text{ where } a, b, c, d \in \mathbb{Z}.$$



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the *size* of the translation taking $\vec{b}$ to $\vec{a}$:

$$p \longmapsto p + (\vec{b} - \vec{a})$$

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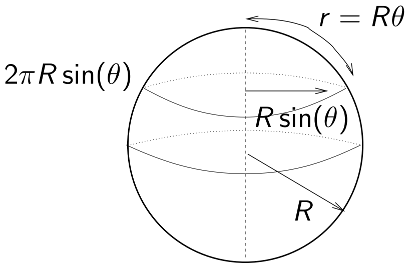
Metric circles on the sphere of radius R.



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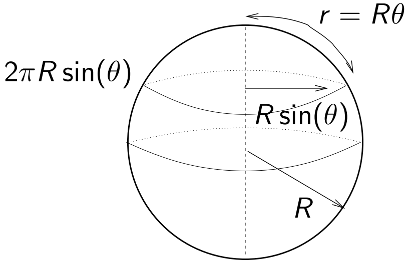

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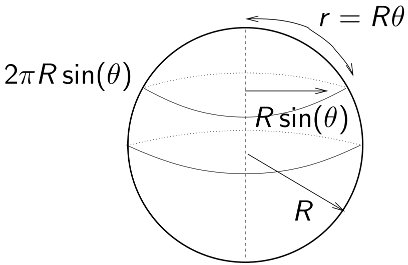

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- As $R \longrightarrow \infty$, the geometry approaches *Euclidean:*

$$C(r) \longrightarrow 2\pi r.$$

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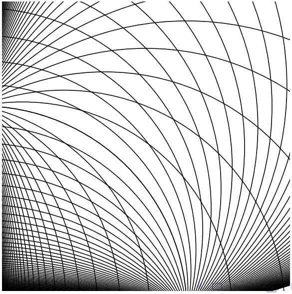

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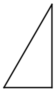
- If the angles α, β, γ in $\triangle$ are π/n, where $n > 0$ is an integer, then the triangles tile the plane.
- If the angles are $\pi/p, \pi/q, \pi/r$ then the three reflections R_1, R_2, R_3 generate a group with presentation with defining relations

$$(R_1)^2 = (R_2)^2 = (R_3)^2 =$$
$$(R_1R_2)^p = (R_2R_3)^q = (R_3R_1)^r = I$$

Tiling $\mathbb{R}^2$ by triangles



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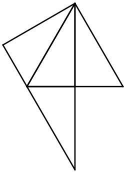

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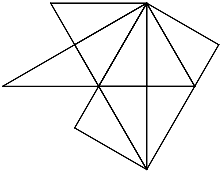

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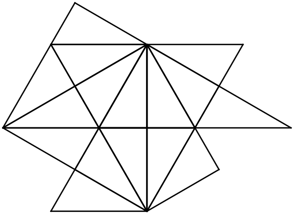

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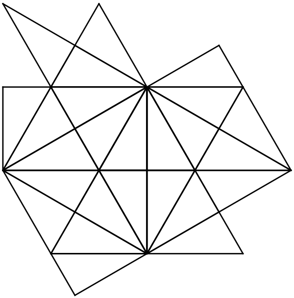

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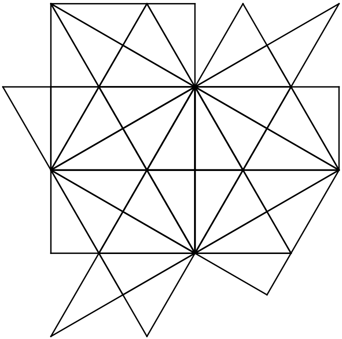

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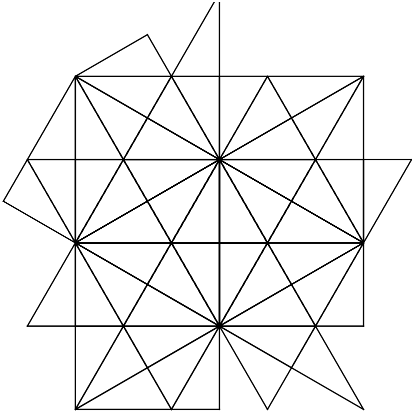

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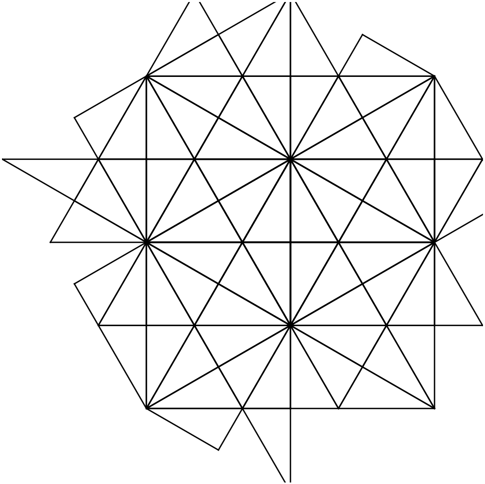

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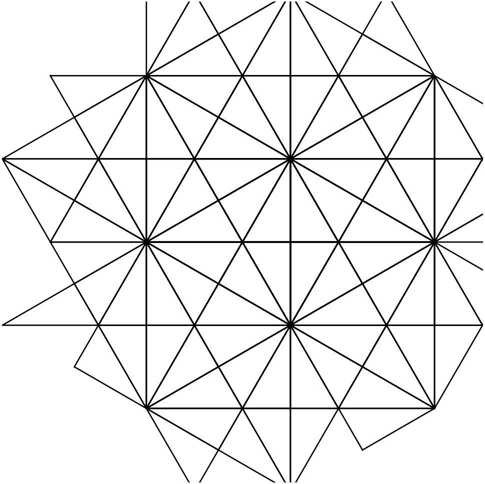

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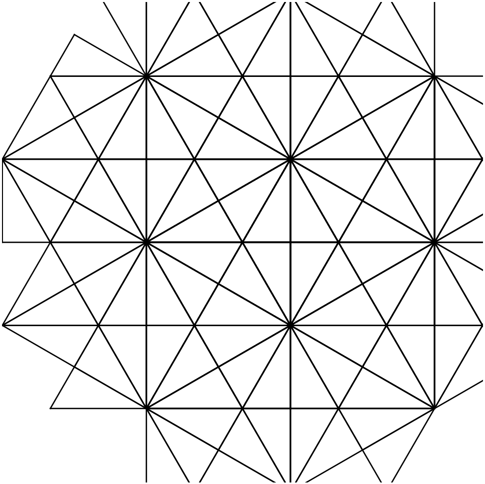

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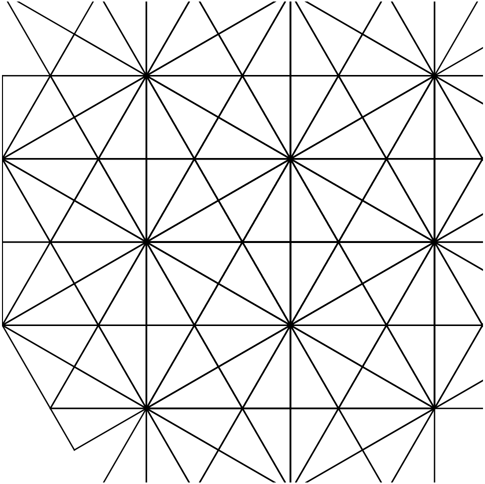

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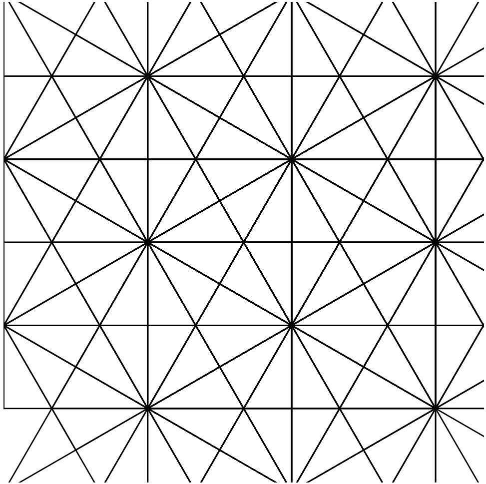

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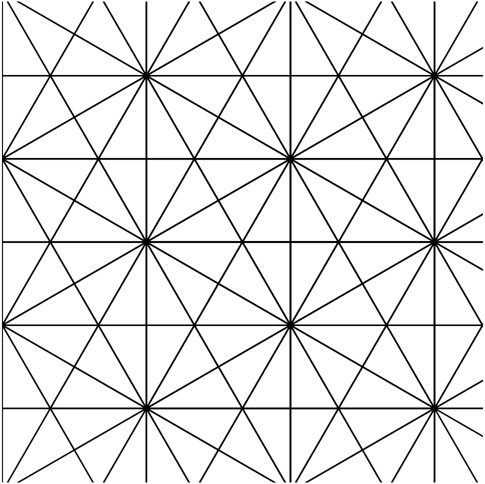

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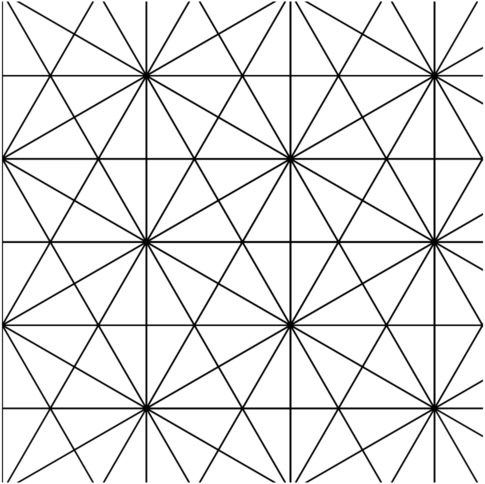

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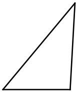

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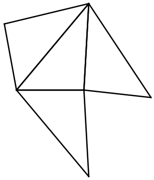

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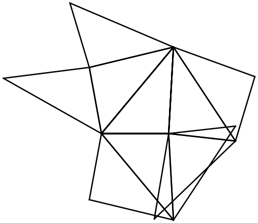

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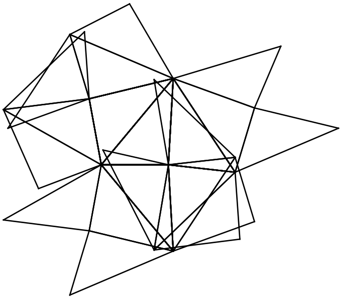

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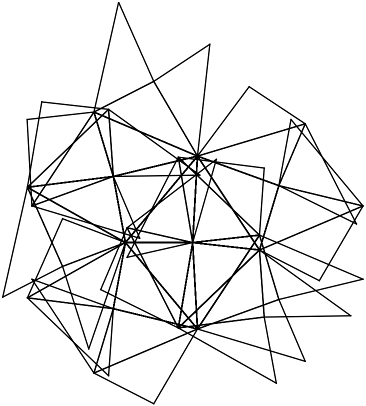

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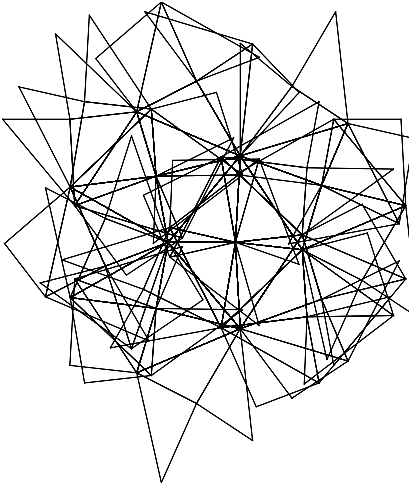

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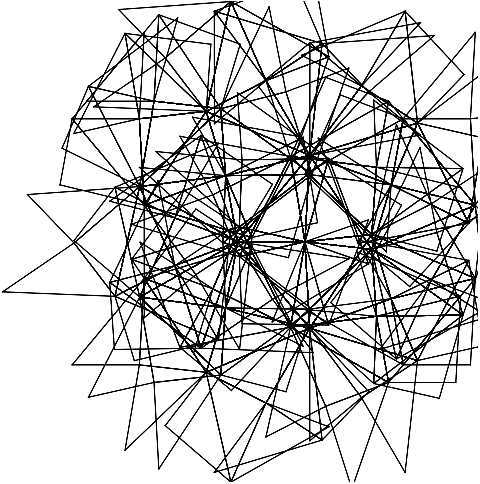
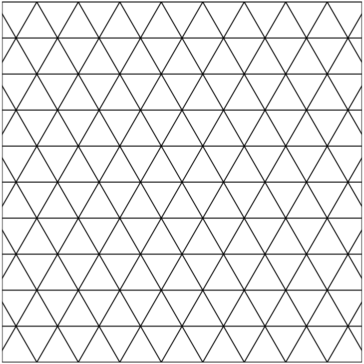


A tiling of the Euclidean plane by equilateral ($\pi/3$-) triangles



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- If $\alpha + \beta + \gamma > \pi$, then a spherical triangle exists with these angles.
  These triangles tile S^2.
  The number of triangles equals

$$\frac{4\pi}{\alpha + \beta + \gamma - \pi},$$

  the numerator equals $\mathrm{area}(S^2)$. and the denominator equals $\mathrm{area}(\triangle)$.

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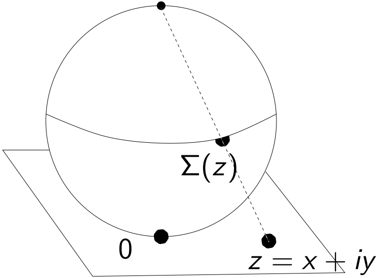
- If $\alpha + \beta + \gamma < \pi$, then a hyperbolic triangle exists with these angles.
  These triangles tile H^2.
- In both spherical and hyperbolic geometry, the angles (α, β, γ) determine $\triangle$ up to *congruence.*

Stereographic Projection



Stereographic projection maps $z = x + iy \in \mathbb{C}$ to

$$\Sigma(z) := \frac{1}{1+|z|^2}\begin{bmatrix} 2z \\ -1+|z|^2 \end{bmatrix}$$

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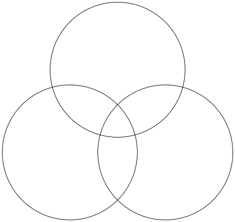
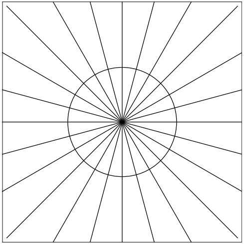
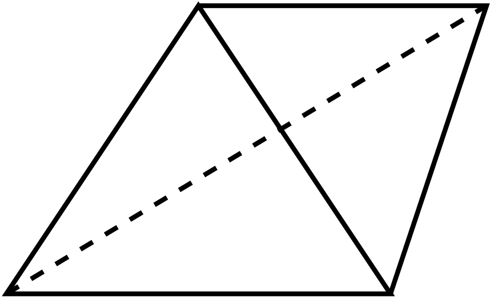


Tiling the sphere by triangles with two right angles



Here is a tiling of S^2 by 24 triangles with angles $\pi/2$, $\pi/2$ and $\pi/6$.

The tetrahedral tiling of the sphere by triangle



Inscribe a tetrahedron in a sphere and then join the centers of its faces to the vertices to obtain a tiling of the sphere by 24 triangles. Each triangle has angles $\pi/2$, $\pi/3$, $\pi/3$.

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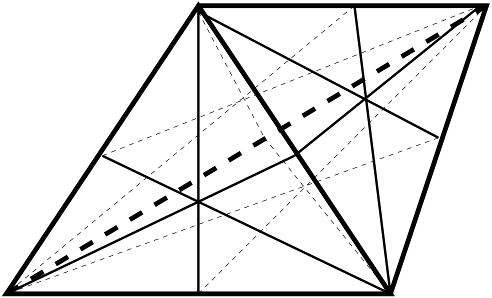

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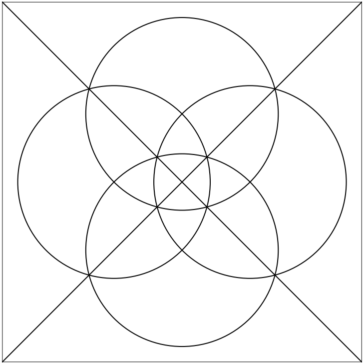

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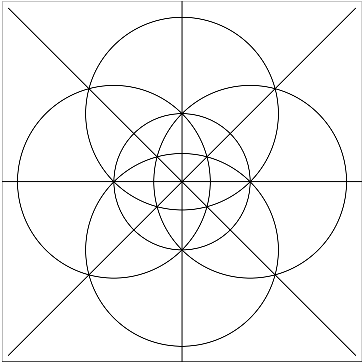
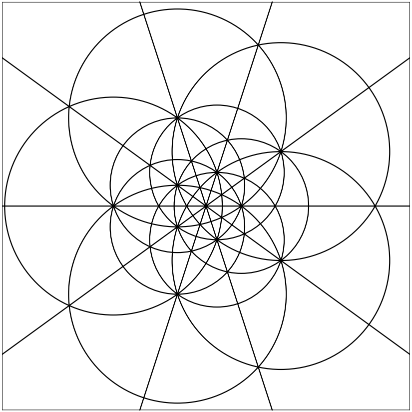


Stereographic projection of the tiling of a sphere by 48 triangles of angles $\pi/2, \pi/3, \pi/4$ corresponding to a regular octahedron inscribed in the sphere.

The Icosahedral Tiling



Tiling the sphere by 120 triangles of angles $\pi/2, \pi/3, \pi/5$ corresponding to an icosahedron.

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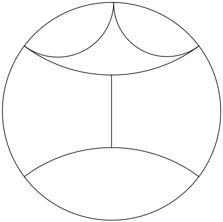

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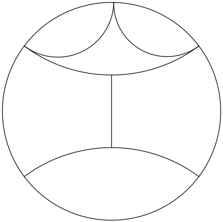
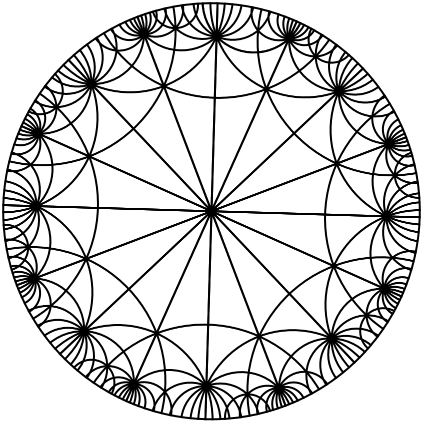
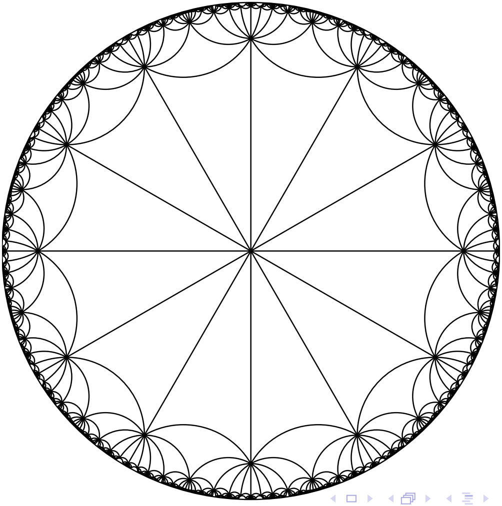


- Given $\alpha, \beta, \gamma \geq 0$ such that $\alpha + \beta + \gamma < \pi$, a unique triangle exists with these angles.

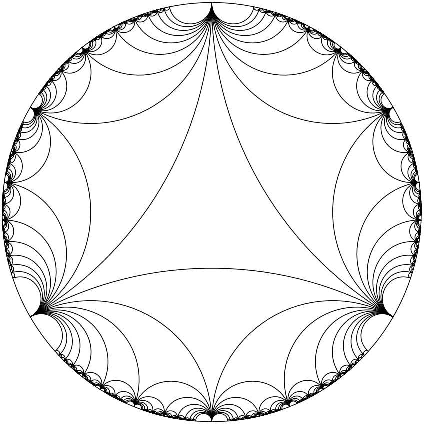
Tiling H^2 by triangles with angles $\pi/2, \pi/4, \pi/8$.



Tiling the hyperbolic plane by $\pi/6$-equilateral triangles



Tiling the hyperbolic plane by triangles with asymptotic sides



Finite area although sides have infinite length.

Matrices as geometric objects

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$$z \overset{\phi}{\longmapsto} \frac{az+b}{cz+d}$$

on $\mathbb{C} \cup \{\infty\}$.

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- Every complex number $a \in \mathbb{C}$ is the trace of some $A \in \mathrm{SL}(2, \mathbb{C})$, for example:

$$A = \begin{bmatrix} a & -1 \\ 1 & 0 \end{bmatrix}.$$

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- Let H^2 be a disc bounded by C_∞.
  C_1, C_2 determine Poincaré geodesics at distance d:

$$\mathrm{tr}(R_1 R_2) = \pm 2\cosh(d).$$

Triangle representations

- If R_1, R_2, R_3 satisfy $(R_i)^2 = I$, then

$$A := R_1 R_2$$
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satisfy $ABC = I$.
- Thus the problem of finding circles intersecting at angles α, β, γ reduces to finding matrices A, B, C satisfying $ABC = I$ and
$$\mathrm{tr}(A) = 2\cos(\alpha)$$
$$\mathrm{tr}(B) = 2\cos(\beta)$$
$$\mathrm{tr}(C) = 2\cos(\gamma).$$

The Lie product

- If A, B, C are found, then R_1, R_2, R_3 can be reconstructed by formulas:

$$\begin{aligned} R_1 &= CA - AC \\ R_2 &= AB - BA \\ R_3 &= BC - CB \end{aligned}$$

  to ensure that $A = R_1 R_2$, etc.
- The *Lie product* $AB - BA$ is analogous to the *cross product* $A \times B$ of vectors.

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- Central to all this theory is the fundamental result characterizing pairs of unimodular 2×2 complex matrices:
- Let $A, B \in \mathrm{SL}(2, \mathbb{C})$, and define $C = (AB)^{-1}$

$$\begin{aligned} a &:= \mathrm{tr}(A) \\ b &:= \mathrm{tr}(B) \\ c &:= \mathrm{tr}(AB) = \mathrm{tr}(C). \end{aligned}$$

Then if $a^2 + b^2 + c^2 - abc \neq 4$, then any other pair (A', B') with the same traces (a, b, c) is conjugate to (A, B).

- If $a^2 + b^2 + c^2 - abc = 4$, then $\exists P$ such that

$$PAP^{-1} = \begin{bmatrix} \alpha & * \\ 0 & 1/\alpha \end{bmatrix}$$
$$PBP^{-1} = \begin{bmatrix} \beta & * \\ 0 & 1/\beta \end{bmatrix}.$$

- so that

$$a = \alpha + 1/\alpha$$
$$b = \beta + 1/\beta$$
$$c = (\alpha\beta) + 1/(\alpha\beta)$$

parametrizing $a^2 + b^2 + c^2 - abc = 4$ by rational functions.

- Conversely, given a, b, c satisfying $a^2 + b^2 + c^2 - abc \neq 4$. Choose γ so that

$$c = \gamma + 1/\gamma.$$

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$$PCP^{-1} = \begin{bmatrix} \gamma & -a\gamma + b \\ 0 & 1/\gamma \end{bmatrix}.$$

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  - Triangles are the building blocks for surfaces.
- Geometry of matrices defines geometric structure on the moduli space.

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  - More complicated Lie groups ($\mathrm{SL}(n, \mathbb{C})$ when $n > 2$).

A (3,3,4)-triangle tiling in the real projective plane
$G = \mathrm{SL}(3, \mathbb{R})$.