Flat Lorentz 3-manifolds and cocompact Fuchsian groups

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1. Introduction

Consider Minkowski 2+1-space $\mathbb{E}$ and let $G \subset \mathrm{SO}(2,1)^0$ be a discrete subgroup. Suppose that a group of affine isometries of $\mathbb{E}$ with linear part G acts properly and freely on $\mathbb{E}$. In a remarkable preprint [**20**], Geoffrey Mess proved the following theorem:

Theorem. *G is not cocompact in* $\mathrm{SO}(2,1)^0$.

Mess deduces this result as part of a general theory of domains of dependence in constant curvature Lorentzian 3-manifolds. We give an alternate proof, using an invariant introduced by Margulis [**18, 19**] and Teichmüller theory.

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2. Background

Let $\mathbb{R}^{2,1}$ be a 3-dimensional real vector space with inner product

$$\mathbb{B}(x,y) = x_1y_1 + x_2y_2 - x_3y_3.$$

The group of linear isometries of $\mathbb{R}^{2,1}$ will be denoted by $\mathrm{SO}(2,1)$. Let $\mathrm{Isom}(\mathbb{R}^{2,1})$ denote the group of *affine isometries*, that is, the group of all transformations of the form

$$\begin{aligned} h : \mathbb{R}^{2,1} &\longrightarrow \mathbb{R}^{2,1} \\ x &\longmapsto g(x) + u \end{aligned}$$

where $g \in \mathrm{O}(2,1)$ and $u \in \mathbb{R}^{2,1}$. We write $g = \mathbb{L}(h)$ and $h = (g,u)$. Evidently $\mathrm{Isom}(\mathbb{R}^{2,1})$ is isomorphic to the semidirect product $\mathrm{O}(2,1) \ltimes \mathbb{R}^{2,1}$ where $\mathbb{R}^{2,1}$ denotes the vector group of *translations* of $\mathbb{E}$.

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Let $G \subset \mathrm{O}(2,1)$ be a subgroup. An *affine deformation of* G is a homomorphism $\phi : G \longrightarrow \mathrm{Isom}(\mathbb{R}^{2,1})$ such that $\mathbb{L}(\phi(g)) = g$. An affine deformation ϕ is *proper* if the resulting action of G by affine transformations on $\mathbb{R}^{2,1}$ is a proper action. Write

$$\phi(g) = (g, u(g)).$$

The condition that ϕ be a homomorphism is that the map $u = u_\phi : G \longrightarrow \mathbb{R}^{2,1}$ satisfy the *cocycle condition*

$$u_\phi(g_1 g_2) = u_\phi(g_1) + g_1 u_\phi(g_2). \tag{1}$$

A map $u : G \longrightarrow \mathbb{R}^{2,1}$ satisfying (1) is called a *cocycle* and the vector space of cocycles is denoted by $Z^1(G, \mathbb{R}^{2,1})$.

If ϕ_1, ϕ_2 are affine deformations of G which are conjugate by translation by $v \in \mathbb{R}^{2,1}$, then the difference $u_{\phi_1} - u_{\phi_2}$ is the cocycle

$$\delta v : g \longmapsto v - g(v).$$

Such a cocycle is called a *coboundary*. The subspace of coboundaries is denoted by $B^1(G, \mathbb{R}^{2,1})$. We say that ϕ_1, ϕ_2 are *translationally conjugate*. Translational conjugacy classes of affine deformations of G correspond to elements in the *cohomology group*

$$H^1(G, \mathbb{R}^{2,1}) = Z^1(G, \mathbb{R}^{2,1}) / B^1(G, \mathbb{R}^{2,1}).$$

Suppose that $\phi : G \longrightarrow \mathrm{Isom}(\mathbb{R}^{2,1})$ is a proper affine deformation. By Fried-Goldman [**11**], the group G is solvable or the linear part

$$\mathbb{L} \circ \phi : G \longrightarrow \mathrm{O}(2,1)$$

is an isomorphism onto a discrete subgroup of $\mathrm{O}(2,1)$. (Indeed, this conclusion is obtained for any proper affine action on $\mathbb{R}^3$.) The solvable groups are easily classified by embedding them as lattices in Lie subgroups which themselves act properly. When G is not solvable, then interesting examples do exist (Margulis [**18, 19**]). Furthermore every *torsionfree non-cocompact* discrete subgroup $G \subset \mathrm{O}(2,1)$ for which $H^1(G; \mathbb{R}^{2,1}) \neq 0$ admits proper affine deformations (Drumm [**8**]).

Recall that an element of $\mathrm{O}(2,1)$ is *hyperbolic* if it has three distinct real eigenvalues. A subgroup $G \subset \mathrm{O}(2,1)$ is *purely hyperbolic* if every element is hyperbolic. A cocompact discrete subgroup contains a purely hyperbolic subgroup of finite index .

3. An invariant of affine isometries

In [**18, 19**], Margulis defines an invariant $\alpha_\phi : G \longrightarrow \mathbb{R}$ of an affine deformation ϕ of a purely hyperbolic subgroup $G \subset \mathrm{O}(2,1)$ as follows. We assume that $G \subset \mathrm{SO}(2,1)^0$. Choose a component $\mathcal{N}_+$ of the complement of 0 in the lightcone. Since any element g of G is hyperbolic its three eigenvalues are distinct positive real numbers

$$\lambda(g) < 1 < \lambda(g)^{-1}.$$

Choose an eigenvector $\mathsf{x}^-(g) \in \mathcal{N}_+$ for $\lambda(g)$ and an eigenvector $\mathsf{x}^+(g) \in \mathcal{N}_+$ for $\lambda(g)^{-1}$, respectively. Then there exists a unique eigenvector $\mathsf{x}^0(g)$ for g with eigenvalue 1 such that:

- $\mathbb{B}(\mathsf{x}^0(g), \mathsf{x}^0(g)) = 1$;
- $(\mathsf{x}^-(g), \mathsf{x}^+(g), \mathsf{x}^0(g))$ is a positively oriented basis.

Notice that $\mathsf{x}^0(g^{-1}) = -\mathsf{x}^0(g)$.

If ϕ is an affine deformation corresponding to a cocycle u, then α_ϕ is defined as:

$$\begin{aligned} \alpha_\phi : G &\longrightarrow \mathbb{R} \\ g &\longmapsto \mathbb{B}(\mathsf{x}^0(g), u(g)). \end{aligned} \tag{2}$$

More generally, $\alpha_\phi(g) = \mathbb{B}(\mathsf{x}^0(g), \phi(g)(x) - x))$ for any $x \in \mathbb{E}$. Furthermore α_ϕ is a class function on G and recently Drumm-Goldman [**10**] have proved that the mapping

$$\begin{aligned} H^1(G, \mathbb{R}^{2,1}) &\longrightarrow \mathbb{R}^G \\ [u] &\longmapsto \alpha_\phi \end{aligned}$$

is injective, that is, α is a complete invariant of the conjugacy class of the affine deformation.

In [**18, 10**], Margulis proved the following theorem (see also Drumm [**7**]):

THEOREM 1 (Margulis). *Suppose that $G \subset \mathrm{SO}(2,1)^0$ is purely hyperbolic and let $\phi : G \longrightarrow \mathrm{Isom}(\mathbb{R}^{2,1})$ be an affine deformation. If there exist $g_1, g_2 \in G$ such that $\alpha_\phi(g_1) > 0 > \alpha_\phi(g_2)$, then ϕ is not proper.*

Affine deformations defining free actions correspond to cocycles for which $\alpha(g) \neq 0$ for $g \neq \mathbb{I}$. We shall say that a cocycle u is *positive* (respectively *negative*) if $\alpha(g) > 0$ (respectively $\alpha(g) < 0$) whenever $\mathbb{I} \neq g \in G$. Clearly u is positive if and only if $-u$ is negative. We conjecture a converse to Theorem 1: *an affine deformation is proper if and only if its cocycle is positive or negative.*

4. Deformation-theoretic interpretation of α

We reduce the proof of Mess's theorem to facts about deformations of hyperbolic Riemann surfaces. Let M be a surface with a complete hyperbolic structure and $\pi = \pi_1(M)$ its fundamental group. A representation $\phi : \pi \longrightarrow \mathrm{SO}(2,1)^0$ is *Fuchsian* if it is an embedding onto a discrete subgroup of $\mathrm{SO}(2,1)^0$. When M is a closed surface, the space of conjugacy classes of Fuchsian representations $\phi : \pi \longrightarrow \mathrm{SO}(2,1)^0$ is an open subset of the space of conjugacy classes of all representations, which identifies with the *Teichmüller space* $\mathfrak{T}(M)$ of M. (See Weil [**26, 27, 28**], §VI of Raghunathan [**22**] for the general theory and Goldman [**12, 13**] for the case of surface groups.) Its tangent space identifies with the cohomology group $H^1(G, \mathbb{R}^{2,1})$ where $G = \phi(\pi)$.

Since the classical theory of Fuchsian groups is usually phrased in terms of $\mathrm{SL}(2,\mathbb{R})$ (rather than $\mathrm{SO}(2,1)$), and since 2×2 matrices are more tractable than 3×3 matrices, we work with $\mathrm{SL}(2,\mathbb{R})$. The Lie groups $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{SO}(2,1)$ are *locally* isomorphic, but not *globally* isomorphic. One model for the local isomorphism is the adjoint representation, as follows. The trace form of any nontrivial representation (for example the Killing form) provides the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ with a Lorentzian inner product invariant under the adjoint representation. Thus $\mathfrak{sl}(2,\mathbb{R})$ is isometric to $\mathbb{R}^{2,1}$; we give an explicit orthogonal basis. In this way the adjoint representation $\mathrm{Ad} : \mathrm{SL}(2,\mathbb{R}) \longrightarrow \mathrm{Isom}(\mathfrak{sl}(2,\mathbb{R}))$ defines a local isomorphism $\rho : \mathrm{SL}(2,\mathbb{R}) \longrightarrow \mathrm{SO}(2,1)$ of Lie groups.

The local isomorphism $\rho : \mathrm{SL}(2,\mathbb{R}) \longrightarrow \mathrm{O}(2,1)$ is not injective — its kernel consists of the center $\{\pm\mathbb{I}\}$ of $\mathrm{SL}(2,\mathbb{R})$. Nor is ρ surjective — its image is the identity component $\mathrm{SO}^0(2,1)$ of $\mathrm{O}(2,1)$. Neither issue is problematic here, since purely hyperbolic discrete subgroups of $\mathrm{SO}(2,1)$ lift to subgroups of $\mathrm{SL}(2,\mathbb{R})$ (Abikoff [**1**], Culler [**6**], Kra [**17**]). Let G be a purely hyperbolic subgroup of $\mathrm{SO}(2,1)$, with inclusion $\iota : G \hookrightarrow \mathrm{SO}(2,1)$. Then there exists a representation $\tilde{\iota} : G \longrightarrow \mathrm{SL}(2,\mathbb{R})$ such that $\iota = \rho \circ \tilde{\iota}$. Furthermore composition with the local isomorphism ρ induces a covering space

$$\mathrm{Hom}(G, \mathrm{SL}(2,\mathbb{R})) \longrightarrow \mathrm{Hom}(G, \mathrm{Isom}^0(\mathbb{R}^{2,1})).$$

Thus smooth paths in $\mathrm{Hom}(G, \mathrm{Isom}^0(\mathbb{R}^{2,1}))$ lift to $\mathrm{Hom}(G, \mathrm{SL}(2,\mathbb{R}))$. Henceforth we suppress $\tilde{\iota}$ (identifying G with its image $\tilde{\iota}(G)$ in $\mathrm{SL}(2,\mathbb{R})$) and consider paths in $\mathrm{Hom}(G, \mathrm{SL}(2,\mathbb{R}))$.

5. $\mathfrak{sl}(2,\mathbb{R})$ and $\mathbb{R}^{2,1}$

For the calculations later, we now give a detailed description of the local isomorphism ρ derived from the adjoint representation.

For convenience, consider the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ with inner prouct

$$\mathbb{B}(X,Y) := \frac{1}{2}\,\mathrm{tr}(XY). \tag{3}$$

The basis

$$e_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, e_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, e_3 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

is orthogonal with respect to $\mathbb{B}$ and satisfies

$$\mathbb{B}(e_1,e_1) = \mathbb{B}(e_2,e_2) = 1, \ \mathbb{B}(e_3,e_3) = -1.$$

This provides an isometry of Lorentzian vector spaces

$$\psi : \mathfrak{sl}(2,\mathbb{R}) \longrightarrow \mathbb{R}^{2,1}$$

$$\begin{bmatrix} v_1 & v_2 \\ v_3 & -v_1 \end{bmatrix} \longmapsto \begin{bmatrix} v_1 \\ (v_2+v_3)/2 \\ (-v_2+v_3)/2 \end{bmatrix}.$$

With respect to this isometry the adjoint representation defines a local isomorphism $\rho : \mathrm{SL}(2,\mathbb{R}) \to \mathrm{O}(2,1)$ satisfying:

$$\psi(\mathrm{Ad}(g)v) = \rho(g)\psi(v)$$

whenever $g \in \mathrm{SL}(2,\mathbb{R})$ and $v \in \mathfrak{sl}(2,\mathbb{R})$. (In other words, $\psi : \mathfrak{sl}(2,\mathbb{R})_{\mathrm{Ad}} \to \mathbb{R}^{2,1}$ is ρ-equivariant.) Explicitly,

$$\mathrm{SL}(2,\mathbb{R}) \xrightarrow{\rho} \mathrm{O}(2,1)$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} 1+2bc & -ac+bd & ac+bd \\ -ab+cd & (a^2-b^2-c^2+d^2)/2 & (-a^2-b^2+c^2+d^2)/2 \\ ab+cd & (-a^2+b^2-c^2+d^2)/2 & (a^2+b^2+c^2+d^2)/2 \end{bmatrix}$$

(where $ad-bc=1$). Differentiation at $\mathbb{I} \in \mathrm{SL}(2,\mathbb{R})$ (that is, at $a=d=1$, $b=c=0$) gives the Lie algebra isomorphism

$$\begin{aligned} \mathfrak{sl}(2,\mathbb{R}) &\longrightarrow \mathfrak{o}(2,1) \\ \begin{bmatrix} v_1 & v_2 \\ v_3 & -v_1 \end{bmatrix} &\longmapsto \begin{bmatrix} 0 & v_3 - v_2 & v_2 + v_3 \\ v_2 - v_3 & 0 & 2v_1 \\ v_2 + v_3 & -2v_1 & 0 \end{bmatrix}. \end{aligned}$$

An element $g \in \mathrm{SL}(2,\mathbb{R})$ is *hyperbolic* if it has two real distinct eigenvalues, which are necessarily reciprocal. If g has eigenvalues μ, μ^{-1} with $|\mu| < 1$, then $\rho(g)$ has eigenvalues $\lambda = \mu^2, 1, \mu^{-2}$. In particular $g \in \mathrm{SL}(2,\mathbb{R})$ is hyperbolic if and only if $\rho(g)$ is hyperbolic. There exists $f \in \mathrm{SL}(2,\mathbb{R})$ such that

$$fgf^{-1} = g_0$$

where

$$g_0 = \pm \begin{bmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{bmatrix}$$

and

$$0 < \mu < 1 < \mu^{-1}.$$

The eigenvectors of $g_0 = \rho(g_0)$ are:

$$\begin{aligned} \mathsf{x}^-(g_0) &= \psi\left(\begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \\ \mathsf{x}^+(g_0) &= \psi\left(\begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \\ \mathsf{x}^0(g_0) &= \psi\left(\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}. \end{aligned}$$

The eigenvectors for g are the images of the eigenvectors of g_0 under f.

Now we derive a formula for $\alpha(g)$ for an affine deformation ϕ which is of the form $h = (\rho(g), \psi(v)(g))$ where $g \in G \subset \mathrm{SL}(2,\mathbb{R})$ and $v \in \mathfrak{sl}(2,\mathbb{R})$. Suppose that $g \in \mathrm{SL}(2,\mathbb{R})$ is hyperbolic. We use the embedding $\mathrm{SL}(2,\mathbb{R}) \hookrightarrow \mathfrak{gl}(2,\mathbb{R})$. Orthogonal projection

$$\begin{aligned} \mathfrak{gl}(2,\mathbb{R}) &\xrightarrow{\Pi} \mathfrak{sl}(2,\mathbb{R}) \\ g &\longmapsto g - \frac{\mathrm{tr}(g)}{2}\mathbb{I} \end{aligned}$$

maps g_0 to a diagonal matrix of trace zero. Dividing $\Pi(g_0)$ by

$$\mathrm{sgn}(\mathrm{tr}(g))\sqrt{-\det(\Pi(g_0))}$$

gives the diagonal matrix corresponding to $\mathsf{x}^0(g_0) \in \mathbb{R}^{2,1}$ (where $\mathrm{sgn}(x)$ denotes the *sign* of a nonzero real number x). Since $\mathrm{tr}(g_0) = \pm(\mu + \mu^{-1})$,

$$\det(\Pi(g_0)) = -(\mu - \mu^{-1})^2 = -\left(\mathrm{tr}(g_0)^2 - 4\right)/4$$

so

$$\begin{aligned}
\operatorname{sgn}(\operatorname{tr}(g_0))\Pi(g_0)/\sqrt{-\det(\Pi(g_0))} \\
&= \operatorname{sgn}(\operatorname{tr}(g_0))\left(g_0 - \frac{\operatorname{tr}(g_0)}{2}\mathbb{I}\right) \Big/ \left(\frac{\sqrt{\operatorname{tr}(g_0)^2-4}}{2}\right) \\
&= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}
\end{aligned}$$

corresponds to $\mathsf{x}^0(g)$. Conjugation by f gives the general formula

$$\psi : \operatorname{sgn}(\operatorname{tr}(g))\left(g - \frac{\operatorname{tr}(g)}{2}\mathbb{I}\right) \Big/ \left(\frac{\sqrt{\operatorname{tr}(g)^2-4}}{2}\right) \longmapsto \mathsf{x}^0(g) \tag{4}$$

From (4) follows a formula for $\alpha(g)$ in terms of traces. Suppose that $G \subset \mathrm{SL}(2,\mathbb{R})$ is purely hyperbolic and $u \in Z^1(G, \mathfrak{sl}(2,\mathbb{R})) \cong Z^1(G, \mathbb{R}^{2,1})$. Taking the trace of the product of (4) with $u(g)$, and applying (2) and (3) yields:

$$\alpha(g) = \operatorname{sgn}(\operatorname{tr}(g)) \frac{\operatorname{tr}(u(g)g)}{\sqrt{\operatorname{tr}(g)^2-4}} \tag{5}$$

6. Trace and displacement length

Let Hyp denote the subset of $\mathrm{SL}(2,\mathbb{R})$ consisting of hyperbolic elements. The image of the trace function $\operatorname{tr} : \mathrm{Hyp} \longrightarrow \mathbb{R}$ consists of the disjoint two intervals $(-\infty, -2)$ and $(2, \infty)$. Furthermore hyperbolic elements $g \in \mathrm{Hyp}$ are determined up to conjugacy by their trace. In terms of hyperbolic geometry, $\operatorname{tr}(g)$ relates to the *displacement length* $\ell(g)$, that is, the minimum distance g moves a point $x \in \mathrm{H}^2_{\mathbb{R}}$. This minimum is realized when x lies in the g-invariant geodesic, which is necessarily unique. Equivalently $\ell(g)$ is the length of the shortest homotopically nontrivial closed curve in the quotient $\mathrm{H}^2_{\mathbb{R}}/\langle g\rangle$. Such a shortest curve is necessarily a simple closed geodesic. Let $\tilde{g} \in \mathrm{SL}(2,\mathbb{R})$ be a lift of $g \in \mathrm{Isom}(\mathbb{R}^{2,1})$ to $\mathrm{SL}(2,\mathbb{R})$, that is, $g = \rho(\tilde{g})$. Displacement length of g relates to $\operatorname{tr}(\tilde{g})$ and the eigenvalue $0 < \mu < 1$ by:

$$\begin{aligned}
\ell(g) &= -2\log\mu \\
|\operatorname{tr}(\tilde{g})| &= 2\cosh(\ell(g)/2)
\end{aligned}$$

(the sign of $\operatorname{tr}(\tilde{g})$ is ambiguous since $\ker(\rho) = \{\pm\mathbb{I}\}$). Since

$$\frac{d|\operatorname{tr}|}{d\ell} = \sinh(\ell/2) > 0 \tag{6}$$

trace depends monotonically on displacement length.

Associated to a cocycle $u \in Z^1(G, \mathbb{R}^{2,1})$ are real analytic paths $\tilde{\iota}_t$ in $\mathrm{Hom}(G, \mathrm{SL}(2,\mathbb{R}))$ of the form

$$\tilde{\iota}_t(g) = g\exp\left(tu(g) + O(t^2)\right)$$

where t is defined in an open interval I_g containing zero. (In general I_g may depend on g.) We say that the cocycle u is *tangent* to the path $\tilde{\iota}_t$.

Given a path $\tilde{\iota}_t \in \mathrm{Hom}(G, \mathrm{SL}(2,\mathbb{R}))$ where $\tilde{\iota}_t(G) \subset \mathrm{Hyp}$, consider the two functions

$$\begin{aligned} \tau_g : I_g &\longrightarrow \mathbb{R} \\ t &\longmapsto \left|\mathrm{tr}\,(\tilde{\iota}_t(g))\right| \end{aligned}$$

and

$$\begin{aligned} L_g : I_g &\longrightarrow \mathbb{R} \\ t &\longmapsto \ell\,(\tilde{\iota}_t(g)) . \end{aligned}$$

When $\tilde{\iota}_t$ corresponds to a path $\mu(t)$ in $\mathfrak{T}(M)$, then $L_g = \ell_g \circ \mu$ where $\ell_g : \mathfrak{T}(M) \to \mathbb{R}$ is the geodesic length function associated to g.

Lemma 2. *Let ϕ be an affine deformation of G corresponding to the cocycle $u \in Z^1(G, \mathbb{R}^{2,1})$ and let $g \in G$. Suppose that $\mu(t)$ is a path in $\mathfrak{T}(M)$ tangent to u. Then*

$$\alpha_\phi(g) = L_g'(0). \tag{7}$$

Furthermore $\alpha_\phi(g)$ and $\tau_g'(0)$ have the same sign.

Proof. Let $\tilde{\iota}_t : G \longrightarrow \mathrm{SL}(2,\mathbb{R})$ be a smooth path of representations starting at the inclusion ι corresponding to $\mu(t)$.

$$\begin{aligned} \tau_g'(0) &= \frac{d}{dt}\Big|_{t=0} |\mathrm{tr}\,\tilde{\iota}_t(g)| \\ &= \pm \frac{d}{dt}\Big|_{t=0} \mathrm{tr}\left(g(\exp(tu(g) + O(t^2)))\right) \\ &= \pm \frac{d}{dt}\Big|_{t=0} \mathrm{tr}\left(g(\mathbb{I} + tu(g) + O(t^2))\right) \\ &= \pm\,\mathrm{tr}\,(gu(g)) \end{aligned}$$

where the sign equals $\mathrm{sgn}(\mathrm{tr}(\tilde{\iota}_t(g))) = \mathrm{sgn}(\mathrm{tr}(\tilde{\iota}_0(g)))$. Applying (5) to the last expression gives

$$\tau_g'(0) = \frac{\sqrt{\mathrm{tr}(g)^2 - 4}}{2}\alpha(g). \tag{8}$$

Thus $\tau_g'(0)$ has the same sign as $\alpha(g)$ as claimed.

To prove (7), apply (6) and the chain rule to obtain:

$$\tau_g'(0) = \sinh\left(\frac{L_g(0)}{2}\right) L_g'(0). \tag{9}$$

Since

$$\sinh\left(\frac{L_g(0)}{2}\right) = \frac{\sqrt{\mathrm{tr}(g)^2 - 4}}{2},$$

(7) follows from (8) and (9). □

Thus a cocycle is positive (respectively negative) in the sense of Theorem 1 if and only if the corresponding deformation in $\mathfrak{T}(M)$ increases (respectively decreases) lengths of closed curves, to first order.

7. Reduction to Teichmüller theory

Suppose that $G \subset \mathrm{SL}(2,\mathbb{R})$ and $\phi : G \longrightarrow \mathrm{Isom}(\mathbb{R}^{2,1})$ is a proper affine deformation. By Theorem 1, the corresponding cocycle $u \in Z^1(G,\mathbb{R}^{2,1})$ is either positive or negative; by replacing u by $-u$ if necessary, we assume that u is positive.

By Fried-Goldman [**11**], G is necessarily discrete and is isomorphic to its image in the group of affine isometries. Suppose that G is cocompact. By passing to a subgroup of finite index, we may assume that G is torsionfree. Then G acts freely on the real hyperbolic plane $\mathbf{H}^2_{\mathbb{R}}$ and since G is discrete and cocompact, $\mathbf{H}^2_{\mathbb{R}}/G$ is a closed hyperbolic surface M. Furthermore G is isomorphic to the fundamental group $\pi_1(M)$. The representation $\tilde{\iota}$ corresponds to a point O in the Teichmüller space $\mathfrak{T}(M)$ and the cohomology class $[u] \in H^1(G,\mathbb{R}^{2,1})$ corresponds to a tangent vector v to $\mathfrak{T}(M)$ at O.

Lemma 3. *There exists a path $\mu(t)$ in $\mathfrak{T}(M)$, defined for all $0 \le t < \infty$ starting at $O \in \mathfrak{T}(M)$ with tangent vector $v \in T_O\mathfrak{T}(M)$:*

$$\mu(0) = O \qquad (10)$$
$$\mu'(0) = v$$

such that, for each $g \in G$, the geodesic length function ℓ_g is convex along $\mu(t)$.

Assuming Lemma 3 and that u is positive, we obtain a contradiction. Since $\alpha(g) > 0$, the directional derivative

$$\mu'(0)\ell_g = v\ell_g = L'_g(0) > 0$$

by Lemma 2. Convexity implies that $\mu'(t)\ell_g$ cannot decrease as $t \longrightarrow +\infty$. Thus

$$(\ell_g \circ \mu)'(t) = \mu'(t)\ell_g \ge \mu'(0)\ell_g = \alpha(g) > 0$$

for all $t \ge 0$. In particular $\ell_g \circ \mu$ is monotone. Furthermore

$$\ell_g(\mu(t)) \longrightarrow +\infty \text{ as } t \longrightarrow +\infty, \qquad (11)$$

that is, *each closed geodesic on the hyperbolic surface μ_t lengthens as $t \longrightarrow +\infty$.*

Such a path μ cannot exist for closed hyperbolic surfaces. Let $N > 0$. Then for only finitely many conjugacy classes $F = \{[g_1], \dots, [g_m]\}$ in $G \cong \pi_1(M)$, the corresponding closed geodesics in M have length $< N$. (Here $[g]$ denotes the conjugacy class of $g \in G$.) For any $g \in G$ with $[g] \notin F$, the length function $L_g(t) > L_g(0) \ge N$. Now consider $[g_i] \in F$. Let

$$\alpha_0 = \min_{1 \le i \le m} \alpha(g_i) > 0.$$

Convexity, together with (7) implies that

$$L_{g_i}(t) \ge L_{g_i}(0) + t\alpha(g_i) \ge t\alpha_0.$$

Hence, for $t > N/\alpha_0$,

$$L_g(t) = \ell_g(\mu_t) > N$$

for all $g \in G - \{\mathbb{I}\}$.

However, for any closed hyperbolic surface M there exists a simple closed geodesic of length at most $2\log(2 - 2\chi(M))$ (Lemma 5.2.1 of Buser [**2**]). Taking $N > 2\log(2 - 2\chi(M))$, we obtain the desired contradiction. ☐

Proof of Lemma 3. Here are two constructions for μ, the first based on the Riemannian geometry of $\mathfrak{T}(M)$ with the Weil-Petersson metric and the second based on Thurston's earthquake flows.

Let $\mu(t)$ be the Weil-Petersson geodesic satisfying (10). By Corollary 4.7 of Wolpert [**30**], the geodesic length function ℓ_g is strictly convex along $\mu(t)$ and the directional derivative $\upsilon\ell_g > 0$, for any $g \in G - \{1\}$. Therefore $\ell_g \circ \mu(t)$ is monotonically increasing for $t > 0$.

However, in general the Weil-Petersson metric is geodesically incomplete (Chu [**5**], Wolpert [**31**]), so that $\mu(t)$ is only defined for $t_1 < t < t_2$ where $t_1 < 0 < t_2$. We show this is impossible under our assumptions on $\mu'(0) = \upsilon$.

By Mumford's compactness theorem (Mumford [**21**], Harvey [**14**], 2.5.1 or Buser [**2**], 6.6.5), the subspace of moduli space consisting of hyperbolic surfaces whose injectivity radius is larger than any positive constant is compact. An incomplete geodesic on a Riemannian manifold must leave every compact set. Therefore, if the Weil-Petersson geodesic $\mu(t)$ cannot be extended to $t_2 < \infty$, then

$$\lim_{t \to t_2} \inf_{g \in G - \{1\}} \ell_g(\mu(t)) = 0,$$

contradicting monotonicity of ℓ_g.

Hence $\mu(t)$ is defined for all $t < \infty$. As above, convexity implies (11).

Alternatively, take μ to be the earthquake path introduced by Thurston (see Kerckhoff [**15, 16**] and Thurston [**24**]). For the given tangent vector υ, there exists a unique measured geodesic lamination λ such that the corresponding earthquake path $\mu(t) = \mathcal{E}_\lambda(t)$ satisfies (10) (Kerckhoff [**16**], Proposition 2.6). By Kerckhoff [**15**] (see also Wolpert [**29**]), each length function ℓ_g is convex along the earthquake path $\mathcal{E}_\lambda$, implying (11). Indeed, ℓ_g is strictly convex along μ since the lamination λ *fills up* M — that is, every nonperipheral simple closed curve σ intersects λ. For otherwise ℓ_σ would be constant along μ, contradicting

$$\left.\frac{d}{dt}\right|_{t=0} \ell_\sigma \circ \mu(t) > 0.$$

□

Remark. Another proof, closer in spirit to the proof in [**20**], involves the density of simple closed curves in the projective measured lamination space. Let $\mathcal{S}$ denote the set of isotopy classes of simple closed curves on M and let $\mathcal{PL}(M)$ denote Thurston's space of projective equivalence classes of measured geodesic laminations on M. Since

$$\begin{aligned} \mathcal{ML}(M) &\longrightarrow T_O\mathfrak{T}(M) \\ \lambda &\longmapsto \mathcal{E}'_\lambda(0) \end{aligned}$$

is a homeomorphism (Proposition 2.6 of [**16**]), there exist $\lambda \in \mathcal{ML}(M)$ satisfying $\mathcal{E}'_\lambda(0) = \upsilon \neq 0$. Theorem 5.1 of [**25**] implies

$$\begin{aligned} \mathcal{PL}(M) &\longrightarrow T^*_O\mathfrak{T}(M) \\ [\lambda] &\longmapsto d\log \ell_\lambda \end{aligned}$$

is an embedding onto a convex sphere in $T^*_O\mathfrak{T}(M)$ (where $\ell_\lambda(N)$ denotes the *length* of the lamination λ as measured in N). Since $\mathcal{S}$ is dense in $\mathcal{PL}(M)$, there exist

$\gamma_1, \gamma_2 \in S$ such that

$$(d \log \ell_{\gamma_1})(\lambda) > 0$$
$$(d \log \ell_{\gamma_2})(\lambda) < 0.$$

Let $g_1, g_2 \in \pi_1(M)$ correspond to γ_1, γ_2 respectively. Then

$$vL_{g_1} > 0, vL_{g_2} < 0,$$

contradicting Theorem 1 and Lemma 2.

REMARK. Mess's original proof uses Lorentzian geometry, and in particular the theory of domains of dependence in constant curvature Lorentzian space forms developed in [**20**] and Scannell [**23**]. As part of his general theory, Mess shows that any affine deformation sufficiently near the holonomy of a complete flat Lorentz 3-manifold is the holonomy of a complete flat Lorentz 3-manifold, that is, the nearby action is also proper and free. The cocycle u corresponds to the velocity vector to an earthquake path $\mathcal{E}_\lambda$ along a measured geodesic lamination λ, and λ is approximated by a *finite measured geodesic lamination*, that is, a disjoint union of simple closed geodesics. However for a finite lamination, the corresponding group action is not free (elements of G corresponding to curves disjoint from λ have fixed points), a contradiction.

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