Locally homogeneous geometric structures

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Dedicated to the memory of Bill Thurston

Outline

1. Geometry through symmetry (Lie, Klein)
2. Projective geometries: deforming 2-dimensional hyperbolic geometry
3. Classification: Moduli spaces of geometric structures $\mathfrak{D}_{(G,X)}(\Sigma)$ associated to topology Σ and homogeneous space $(G, X = G/H)$
4. Examples: Euclidean, hyperbolic geometry
5. Examples: Real, complex projective geometry
6. Examples: Minkowski space, Anti-de Sitter space
7. Moduli of surface group representations (higher Teichmüller theory)
8. Classification of complete affine 3-manifolds
9. Margulis spacetimes, crooked geometry

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  - and then by discrete groups which don't act properly.

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- (Ehresmann 1936): Geometric manifold M modeled on X.

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- (Thurston 1976): 3-manifolds canonically decompose into *locally homogeneous Riemannian* pieces (8 types). (proved by Perelman)

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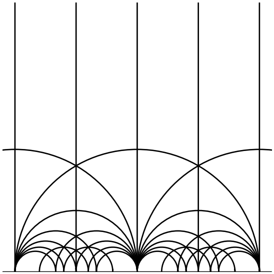

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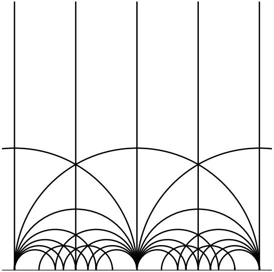

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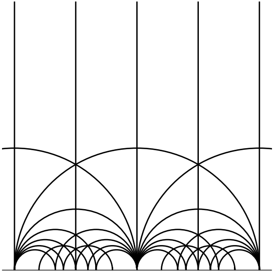
  - *Example:* The 2-torus admits a *moduli space* of Euclidean structures.



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  - Cartan connections exist on fiber bundles and an Ehresmann structure determines a flat connection on this fiber bundle with a canonical section describing the local coordinates.
  - *Local* deformation theory of geometric structures $\Longleftrightarrow$ *local* deformation theory of flat connections — representations of $\pi_1(\Sigma)$.

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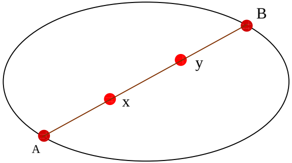

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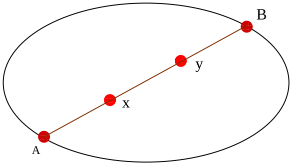

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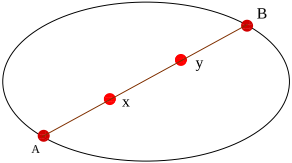
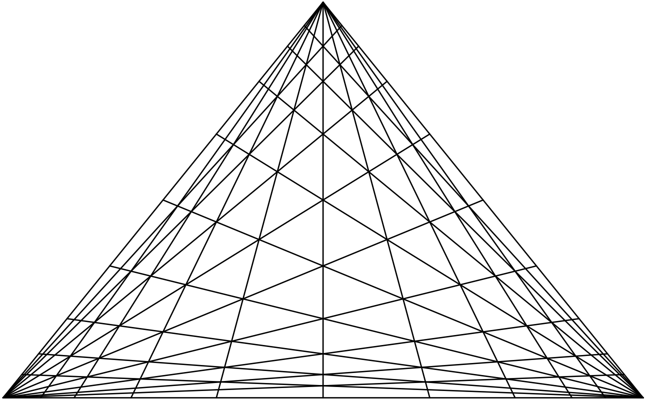


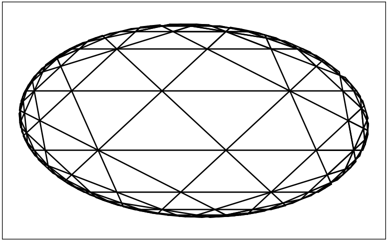
- Projective geometry *contains* hyperbolic geometry.
  - Hyperbolic structures *are* convex $\mathbb{RP}^n$-structures.

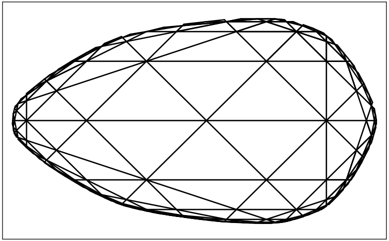
Projective deformation of equilateral 60^o-triangle tiling



This tesselation of the open triangular region in $\mathbb{RP}^2$ is equivalent to the tiling of the Euclidean plane by equilateral triangles.

Example: Projective deformation of a hyperbolic tiling





Both domains are tiled by 60^o,60^o,45^o)-triangles, invariant under a Coxeter group $\Gamma(3, 3, 4)$. First is bounded by a conic (hyperbolic geometry). Second is invariant under Weyl group associated to

$$\begin{bmatrix} 2 & -1 & -2 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

with domain bounded by $C^{1+\alpha}$-convex curve where $0 < \alpha < 1$.

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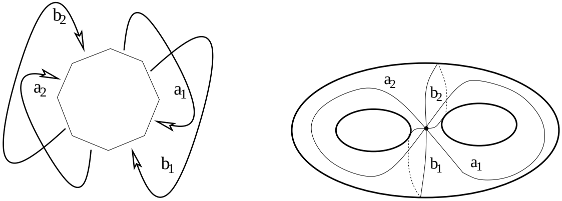

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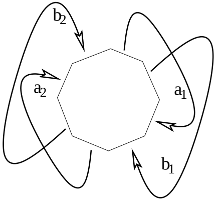
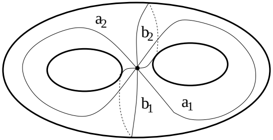
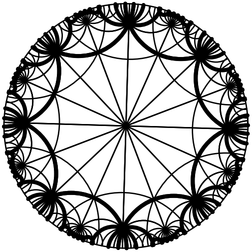
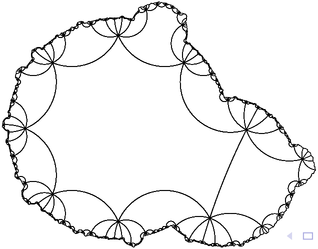




- Realize these identifications isometrically for a regular 45^o-octagon.



Example: Quasi-Fuchsian $\mathbb{CP}^1$-structure



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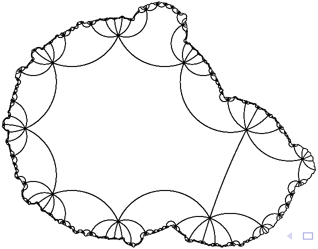
- Start with a hyperbolic structure on a surface Σ represented as a quotient H^2/Γ_0 where $\Gamma_0 \subset \mathsf{PSL}(2,\mathbb{R}) = \mathsf{Isom}^+(\mathsf{H}^2)$. Regard H^2 as an open hemisphere in $\mathbb{CP}^1$ invariant under $\mathsf{PSL}(2,\mathbb{R})$.



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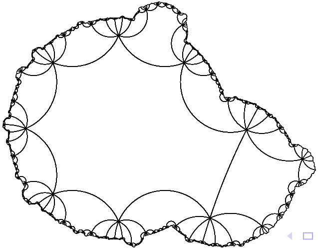

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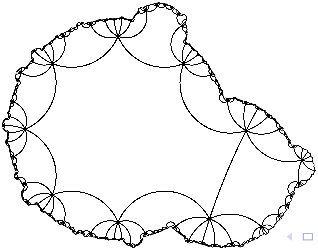

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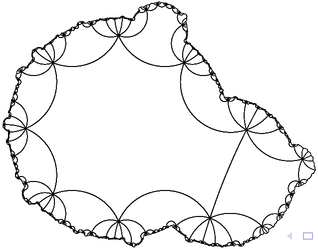
  - Hyperbolic structure $\Longrightarrow$ $\mathbb{CP}^1$-structure.
- Deform the representation of Γ_0 in $\mathsf{PSL}(2,\mathbb{C}) \supset \mathsf{PSL}(2,\mathbb{R})$.
- For Γ_t sufficiently near Γ_0, the deformation Γ_t arises from an embedding of Γ_0 as a discrete group acting properly an open subset $\Omega \subset \mathbb{CP}^1$. Unless Γ_t is Fuchsian, $\partial\Omega$ is a fractal Jordan curve of Hausdorff dimension > 1 (Bowen).



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    - $\mathrm{AdS}^n \cong \mathrm{O}(n-1,2)/\big(\mathrm{O}(n-2)\times \mathrm{O}(1,2)\big)$.
- Let $H = \mathrm{PSL}(2,\mathbb{R})$. For $n=3$, anti-de Sitter geometry identifies with $G = H \times H$ acting (isometrically) on $X = H$ by *left- and right-multiplication:*

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Example: Anti-de Sitter structures

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- A closed 3-dimensional *AdS*-manifold is a quotient $X/\mathrm{graph}(\rho)$ where $\Gamma \subset H$ is a cocompact lattice and

$$\mathrm{graph}(\rho) = \{\big(\gamma, \rho(\gamma)\big) \mid \gamma \in \Gamma\}$$

is the *graph* of ρ. (Kulkarni-Raymond 1985)

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- *Mapping class group*

$$\mathrm{Mod}(\Sigma) := \pi_0\big(\mathrm{Diff}(\Sigma)\big)$$

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  - For general split real forms, these *Hitchin representations* are discrete embeddings (Labourie 2005) and correspond to geometric structures on compact manifolds (Guichard-Wienhard 2011).

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Complete flat Lorentz manifolds

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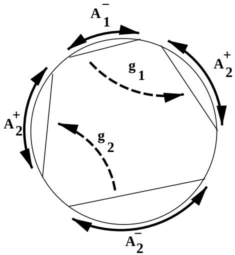
Ping-pong in H^2



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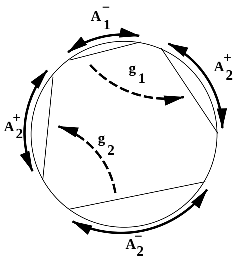

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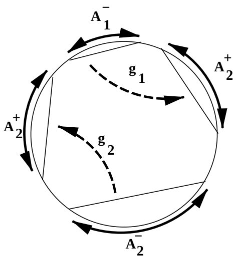

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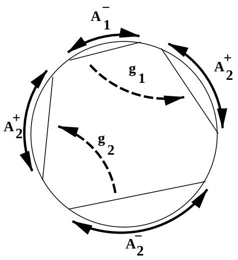
  - paired by isometries $\mathfrak{h}_i^- \xrightarrow{g_i} H^2 \setminus \mathfrak{h}_i^+$.

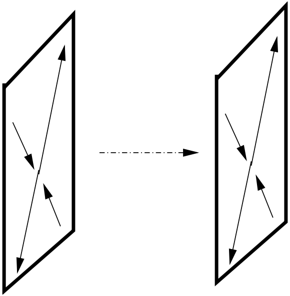
Ping-pong in H^2



- Start with mutually disjoint halfplanes $\mathfrak{h}_1^-, \mathfrak{h}_1^+, \ldots, \mathfrak{h}_n^-, \mathfrak{h}_n^+$
  - paired by isometries $\mathfrak{h}_i^- \xrightarrow{g_i} H^2 \setminus \mathfrak{h}_i^+$.
- $g_1, \ldots, g_n$ *freely* generate group with fundamental domain

$$H^2 \setminus \bigcup_{i=1}^{n} \mathfrak{h}_i^{\pm}.$$

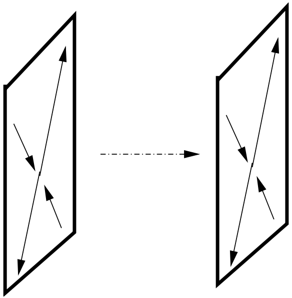
Cyclic groups



A boost identifying two parallel planes

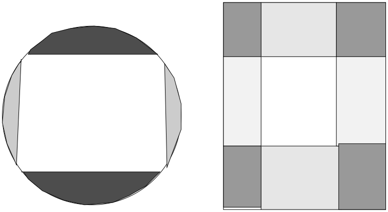
Cyclic groups

- Most elements $\gamma \in \Gamma$ are *boosts*, affine deformations of hyperbolic elements of $SO(2,1)$. A fundamental domain is the *parallel slab* bounded by two parallel planes.



A boost identifying two parallel planes

Parallel slabs don't work!



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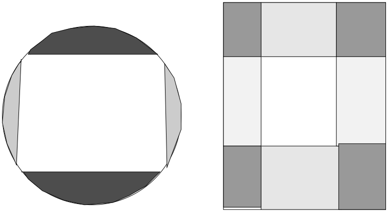

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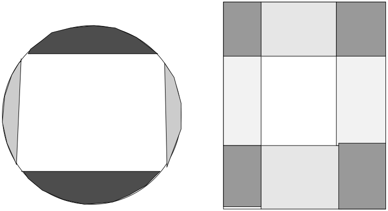

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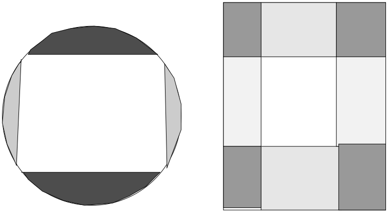

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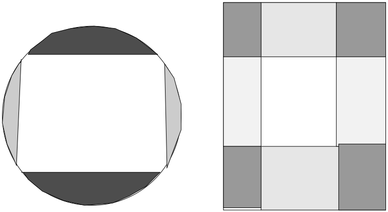

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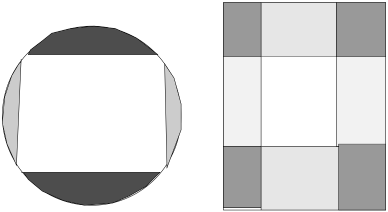

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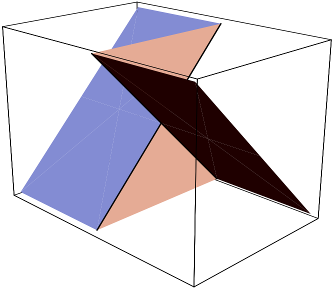
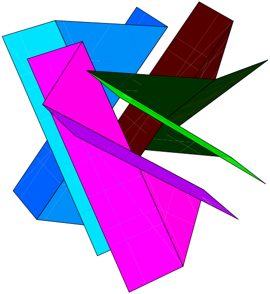
- Complements of *parallel* slabs always intersect,
- Unsuitable for building Schottky groups!

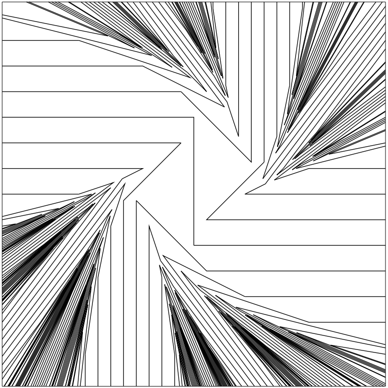
Drumm's Schottky groups

The classical construction of Schottky groups fails using affine half-spaces and slabs. Drumm's geometric construction uses *crooked planes*, PL hypersurfaces adapted to the Lorentz geometry which bound fundamental polyhedra for Schottky groups.





Proper affine deformation of level 2 congruence subgroup of $GL(2,\mathbb{Z})$



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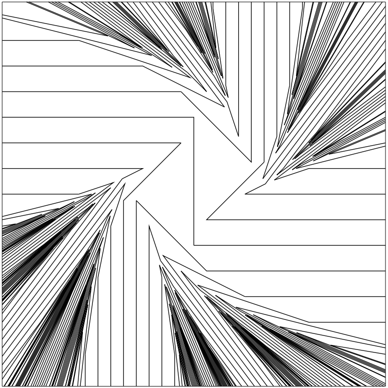

Proper affine deformations exist even for *lattices* (Drumm).

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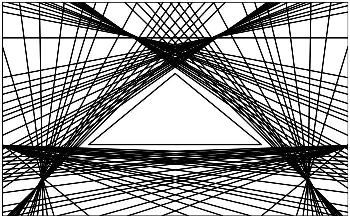
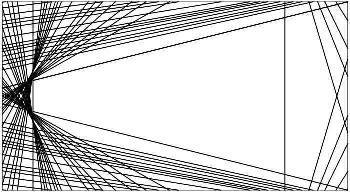
- (2012) Choi and Danciger-Guéritaud-Kassel have announced, independently, quite different proofs of *Topological Tameness:* Every nonsolvable complete flat affine 3-manifold (Margulis spacetime) is homeomorphic to a solid handlebody.

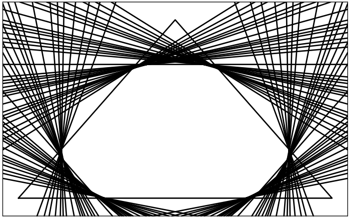
Deformation spaces for surfaces with $\chi(\Sigma)$



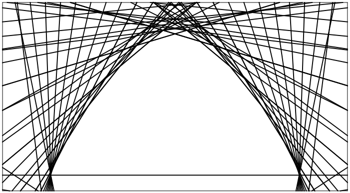
(k) Three-holed sphere



(l) Two-holed $\mathbb{RP}^2$



(m) One-holed torus



(n) One-holed Klein bottle

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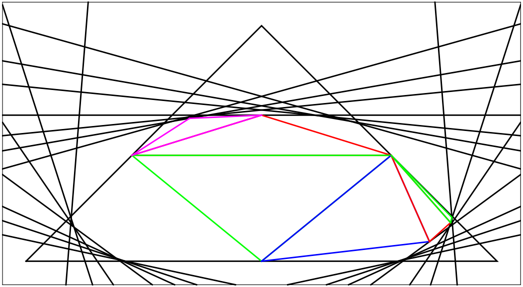
- Birman-Series argument $\Longrightarrow$ For 1-holed torus, these points of strict convexity have Hausdorff dimension zero.

Realizing an ideal triangulation of the one-holed torus by crooked planes



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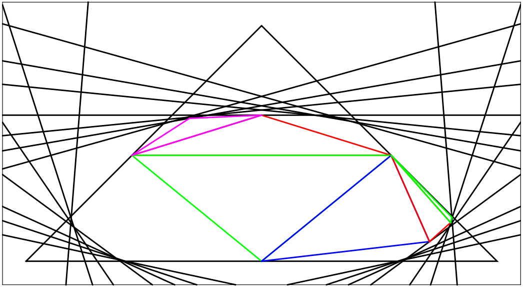

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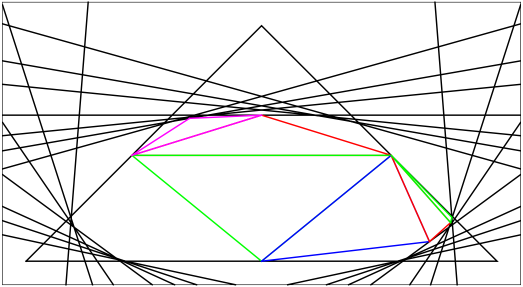

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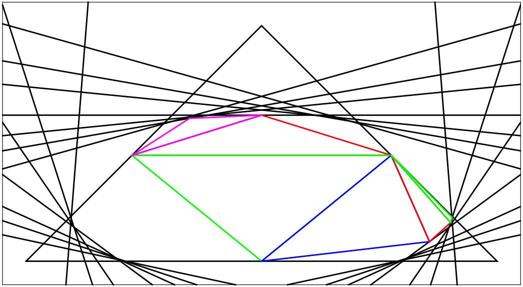

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- Flip of ideal triangulation $\longleftrightarrow$ moving to adjacent triangle.