Geometry and Dynamics of Surface Group Representations

William M. Goldman

Department of Mathematics University of Maryland

Colloque de mathématiques de Montréal
CRM/ISM
14 September 2007

Geometry through symmetry

In his 1872 *Erlangen Program*, Felix Klein proposed that a *geometry* is the study of properties of an abstract space X which are invariant under a transitive group G of transformations of X.

Library of Congress

Putting geometric structure on a topological space

- *Topology:* Smooth manifold Σ with coordinate patches U_α;
- Charts — *diffeomorphisms*

$$U_\alpha \xrightarrow{\psi_\alpha} \psi_\alpha(U_\alpha) \subset X$$

- On components of $U_\alpha \cap U_\beta$, $\exists g \in G$ such that

$$g \circ \psi_\alpha = \psi_\beta..$$

- Local (G, X)-geometry independent of patch.
- (Ehresmann 1936): Geometric manifold M modeled on X.

Putting geometric structure on a topological space

- *Topology:* Smooth manifold Σ with coordinate patches U_α;
- Charts — *diffeomorphisms*

$$U_\alpha \xrightarrow{\psi_\alpha} \psi_\alpha(U_\alpha) \subset X$$

- On components of $U_\alpha \cap U_\beta$, $\exists g \in G$ such that

$$g \circ \psi_\alpha = \psi_\beta..$$

- Local (G, X)-geometry independent of patch.
- (Ehresmann 1936): Geometric manifold M modeled on X.

Putting geometric structure on a topological space

- *Topology:* Smooth manifold Σ with coordinate patches U_α;
- Charts — *diffeomorphisms*

$$U_\alpha \xrightarrow{\psi_\alpha} \psi_\alpha(U_\alpha) \subset X$$

- On components of $U_\alpha \cap U_\beta$, $\exists g \in G$ such that

$$g \circ \psi_\alpha = \psi_\beta..$$

- Local (G, X)-geometry independent of patch.
- (Ehresmann 1936): Geometric manifold M modeled on X.

Putting geometric structure on a topological space

- *Topology:* Smooth manifold Σ with coordinate patches U_α;
- Charts — *diffeomorphisms*

$$U_\alpha \xrightarrow{\psi_\alpha} \psi_\alpha(U_\alpha) \subset X$$

- On components of $U_\alpha \cap U_\beta$, $\exists g \in G$ such that

$$g \circ \psi_\alpha = \psi_\beta..$$

- Local (G, X)-geometry independent of patch.
- (Ehresmann 1936): Geometric manifold M modeled on X.

Putting geometric structure on a topological space

- *Topology:* Smooth manifold Σ with coordinate patches U_α;
- Charts — *diffeomorphisms*

$$U_\alpha \xrightarrow{\psi_\alpha} \psi_\alpha(U_\alpha) \subset X$$

- On components of $U_\alpha \cap U_\beta$, $\exists g \in G$ such that

$$g \circ \psi_\alpha = \psi_\beta..$$

- Local (G, X)-geometry independent of patch.
- (Ehresmann 1936): Geometric manifold M modeled on X.

Putting geometric structure on a topological space

- *Topology:* Smooth manifold Σ with coordinate patches U_α;
- Charts — *diffeomorphisms*

$$U_\alpha \xrightarrow{\psi_\alpha} \psi_\alpha(U_\alpha) \subset X$$

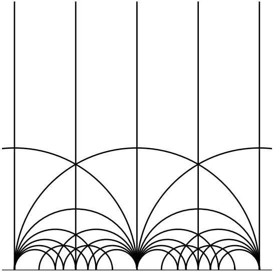
- On components of $U_\alpha \cap U_\beta$, $\exists g \in G$ such that

$$g \circ \psi_\alpha = \psi_\beta..$$

- Local (G, X)-geometry independent of patch.
- (Ehresmann 1936): Geometric manifold M modeled on X.

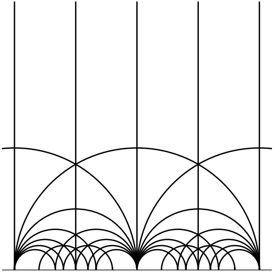
Classfication of geometric structures

- *Basic question:* Given a *topology* Σ and a *geometry* $X = G/H$, determine all possible ways of providing Σ with the local geometry of (X, G).
- *Example:* The 2-sphere does not admit Euclidean-geometry structure: $\nexists$ metrically accurate world atlas.
- *Example:* The 2-torus admits a rich *moduli space* of Euclidean-geometry structures.



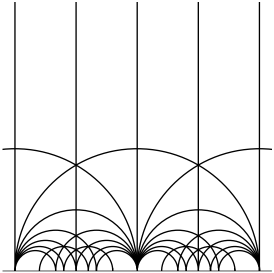
Classfication of geometric structures

- *Basic question:* Given a *topology* Σ and a *geometry* $X = G/H$, determine all possible ways of providing Σ with the local geometry of (X, G).
- *Example:* The 2-sphere does not admit Euclidean-geometry structure: $\nexists$ metrically accurate world atlas.
- *Example:* The 2-torus admits a rich *moduli space* of Euclidean-geometry structures.



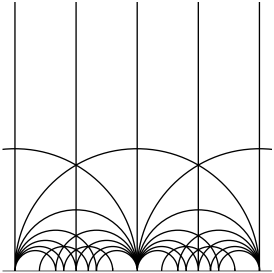
Classfication of geometric structures

- *Basic question:* Given a *topology* Σ and a *geometry* $X = G/H$, determine all possible ways of providing Σ with the local geometry of (X, G).
- *Example:* The 2-sphere does not admit Euclidean-geometry structure: $\nexists$ metrically accurate world atlas.
- *Example:* The 2-torus admits a rich *moduli space* of Euclidean-geometry structures.



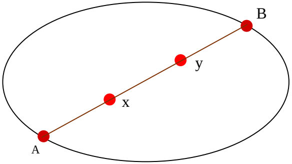
Classfication of geometric structures

- *Basic question:* Given a *topology* Σ and a *geometry* $X = G/H$, determine all possible ways of providing Σ with the local geometry of (X, G).
- *Example:* The 2-sphere does not admit Euclidean-geometry structure: $\nexists$ metrically accurate world atlas.
- *Example:* The 2-torus admits a rich *moduli space* of Euclidean-geometry structures.



Quotients of domains

- Suppose that $\Omega \subset X$ is an open subset invariant under a subgroup $\Gamma \subset G$ such that:
  - Γ is discrete;
  - Γ acts *properly* and *freely* on Ω
- Then $M = \Omega/\Gamma$ is a (G, X)-manifold covered by Ω.
- *Convex* $\mathbb{RP}^n$*-structures:* $\Omega \subset \mathbb{RP}^n$ convex domain.
- For example, the projective geometry of the interior of a quadric Ω is *hyperbolic geometry.*

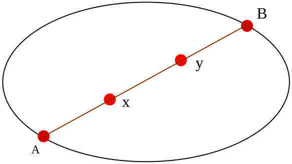


- The hyperbolic distance is defined by cross-ratios:

$$d(x, y) = \log[A, x, y, B]$$

Quotients of domains

- Suppose that $\Omega \subset X$ is an open subset invariant under a subgroup $\Gamma \subset G$ such that:
  - Γ is discrete;
  - Γ acts *properly* and *freely* on Ω
- Then $M = \Omega/\Gamma$ is a (G, X)-manifold covered by Ω.
- *Convex $\mathbb{RP}^n$-structures:* $\Omega \subset \mathbb{RP}^n$ convex domain.
- For example, the projective geometry of the interior of a quadric Ω is *hyperbolic geometry.*

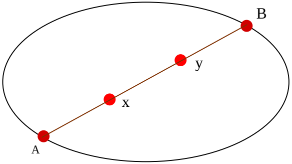


- The hyperbolic distance is defined by cross-ratios:

$$d(x, y) = \log[A, x, y, B]$$

Quotients of domains

- Suppose that $\Omega \subset X$ is an open subset invariant under a subgroup $\Gamma \subset G$ such that:
  - Γ is discrete;
  - Γ acts *properly* and *freely* on Ω
- Then $M = \Omega/\Gamma$ is a (G, X)-manifold covered by Ω.
- *Convex* $\mathbb{RP}^n$*-structures:* $\Omega \subset \mathbb{RP}^n$ convex domain.
- For example, the projective geometry of the interior of a quadric Ω is *hyperbolic geometry.*

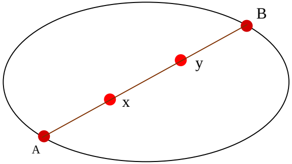


- The hyperbolic distance is defined by cross-ratios:

$$d(x, y) = \log[A, x, y, B]$$

Quotients of domains

- Suppose that $\Omega \subset X$ is an open subset invariant under a subgroup $\Gamma \subset G$ such that:
  - Γ is discrete;
  - Γ acts *properly* and *freely* on Ω
- Then $M = \Omega/\Gamma$ is a (G, X)-manifold covered by Ω.
- *Convex* $\mathbb{RP}^n$*-structures:* $\Omega \subset \mathbb{RP}^n$ convex domain.
- For example, the projective geometry of the interior of a quadric Ω is *hyperbolic geometry.*

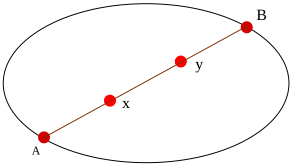


- The hyperbolic distance is defined by cross-ratios:

$$d(x, y) = \log[A, x, y, B]$$

Quotients of domains

- Suppose that $\Omega \subset X$ is an open subset invariant under a subgroup $\Gamma \subset G$ such that:
  - Γ is discrete;
  - Γ acts *properly* and *freely* on Ω
- Then $M = \Omega/\Gamma$ is a (G, X)-manifold covered by Ω.
- *Convex $\mathbb{RP}^n$-structures:* $\Omega \subset \mathbb{RP}^n$ convex domain.
- For example, the projective geometry of the interior of a quadric Ω is *hyperbolic geometry.*

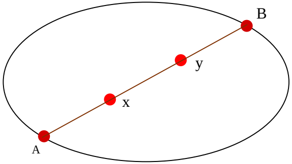


- The hyperbolic distance is defined by cross-ratios:

$$d(x, y) = \log[A, x, y, B]$$

Quotients of domains

- Suppose that $\Omega \subset X$ is an open subset invariant under a subgroup $\Gamma \subset G$ such that:
  - Γ is discrete;
  - Γ acts *properly* and *freely* on Ω
- Then $M = \Omega/\Gamma$ is a (G, X)-manifold covered by Ω.
- *Convex* $\mathbb{RP}^n$*-structures:* $\Omega \subset \mathbb{RP}^n$ convex domain.
- For example, the projective geometry of the interior of a quadric Ω is *hyperbolic geometry.*

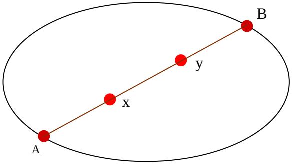


- The hyperbolic distance is defined by cross-ratios:

$$d(x, y) = \log[A, x, y, B]$$

Quotients of domains

- Suppose that $\Omega \subset X$ is an open subset invariant under a subgroup $\Gamma \subset G$ such that:
  - Γ is discrete;
  - Γ acts *properly* and *freely* on Ω
- Then $M = \Omega/\Gamma$ is a (G, X)-manifold covered by Ω.
- *Convex $\mathbb{RP}^n$-structures:* $\Omega \subset \mathbb{RP}^n$ convex domain.
- For example, the projective geometry of the interior of a quadric Ω is *hyperbolic geometry.*

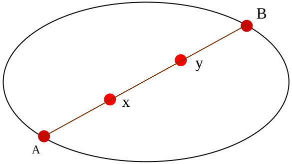


- The hyperbolic distance is defined by cross-ratios:

$$d(x, y) = \log[A, x, y, B]$$

Quotients of domains

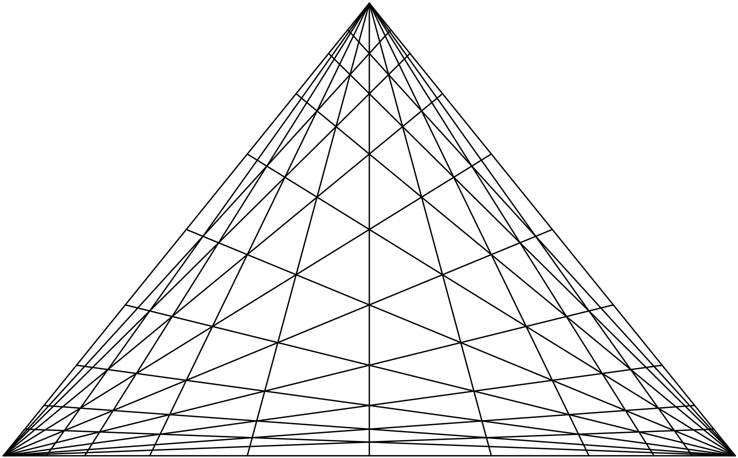
- Suppose that $\Omega \subset X$ is an open subset invariant under a subgroup $\Gamma \subset G$ such that:
  - Γ is discrete;
  - Γ acts *properly* and *freely* on Ω
- Then $M = \Omega/\Gamma$ is a (G, X)-manifold covered by Ω.
- *Convex* $\mathbb{RP}^n$*-structures:* $\Omega \subset \mathbb{RP}^n$ convex domain.
- For example, the projective geometry of the interior of a quadric Ω is *hyperbolic geometry.*



- The hyperbolic distance is defined by cross-ratios:

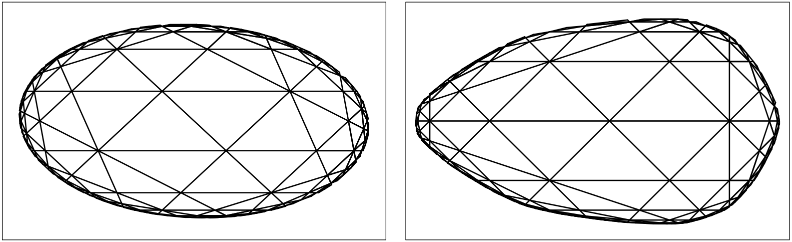
$$d(x, y) = \log[A, x, y, B]$$

Example: A projective tiling by equilateral 60^o-triangles



This tesselation of the open triangular region is equivalent to the tiling of the Euclidean plane by equilateral triangles.

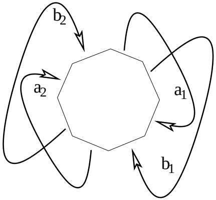
Example: A projective deformation of a tiling of the hyperbolic plane by $(60^o, 60^o, 45^o)$-triangles.

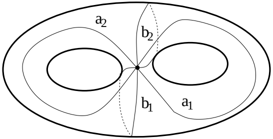


Both domains are tiled by triangles, invariant under a Coxeter group $\Gamma(3,3,4)$. The first domain is bounded by a conic and enjoys hyperbolic geometry. The second domain is bounded by $C^{1+\alpha}$-convex curve where $0 < \alpha < 1$.

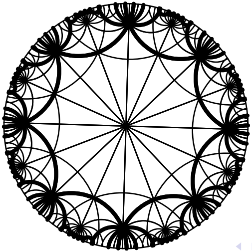
Example: A hyperbolic structure on a surface of genus two

- Identify sides of an octagon to form a closed genus two surface.



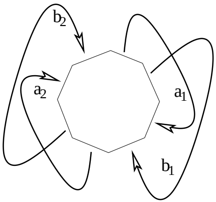


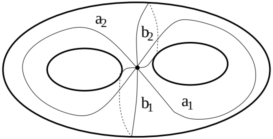
- Realize these identifications isometrically for a regular 45^o-octagon.



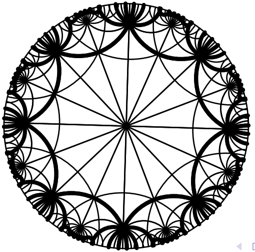
Example: A hyperbolic structure on a surface of genus two

- Identify sides of an octagon to form a closed genus two surface.



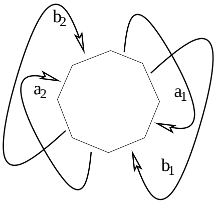


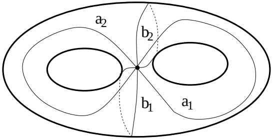
- Realize these identifications isometrically for a regular 45^o-octagon.



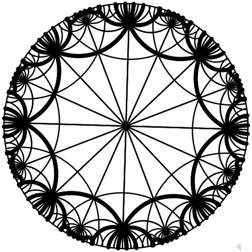
Example: A hyperbolic structure on a surface of genus two

- Identify sides of an octagon to form a closed genus two surface.





- Realize these identifications isometrically for a regular 45^o-octagon.



Modeling structures on representations of π_1

- *Marked* (G,X)-*structure* on Σ: diffeomorphism $\Sigma \xrightarrow{f} M$ where M is a (G,X)-manifold.
- Marked (G,X)-structures (f_i, M_i) are *isotopic* $\Longleftrightarrow$ $\exists$ isomorphism $M_1 \xrightarrow{\phi} M_2$ with $\phi \circ f_1 \simeq f_2$.
- Define the *deformation space*

$$\mathfrak{D}_{(G,X)}(\Sigma) := \left\{ \text{Marked } (G,X)\text{-structures on } \Sigma \right\} / \text{Isotopy}$$

- $\eta \in \text{Diff}(\Sigma)$ acts on marked (G,X)-structures: $(f, M) \longmapsto (f \circ \eta, M)$.
- *Mapping class group*

$$\text{Mod}(\Sigma) := \pi_0(\text{Diff}(\Sigma))$$

acts on $\mathfrak{D}_{(G,X)}(\Sigma)$.

Modeling structures on representations of π_1

- *Marked* (G,X)*-structure* on Σ: diffeomorphism $\Sigma \xrightarrow{f} M$ where M is a (G,X)-manifold.
- Marked (G,X)-structures (f_i, M_i) are *isotopic* $\iff \exists$ isomorphism $M_1 \xrightarrow{\phi} M_2$ with $\phi \circ f_1 \simeq f_2$.
- Define the *deformation space*

$$\mathfrak{D}_{(G,X)}(\Sigma) := \left\{ \text{Marked } (G,X)\text{-structures on } \Sigma \right\} / \text{Isotopy}$$

- $\eta \in \mathrm{Diff}(\Sigma)$ acts on marked (G,X)-structures: $(f, M) \longmapsto (f \circ \eta, M)$.
- *Mapping class group*

$$\mathrm{Mod}(\Sigma) := \pi_0(\mathrm{Diff}(\Sigma))$$

acts on $\mathfrak{D}_{(G,X)}(\Sigma)$.

Modeling structures on representations of π_1

- *Marked* (G, X)*-structure* on Σ: diffeomorphism $\Sigma \xrightarrow{f} M$ where M is a (G, X)-manifold.
- Marked (G, X)-structures (f_i, M_i) are *isotopic* $\iff \exists$ isomorphism $M_1 \xrightarrow{\phi} M_2$ with $\phi \circ f_1 \simeq f_2$.
- Define the *deformation space*

$$\mathfrak{D}_{(G,X)}(\Sigma) := \left\{ \text{Marked } (G, X)\text{-structures on } \Sigma \right\} / \text{Isotopy}$$

- $\eta \in \mathrm{Diff}(\Sigma)$ acts on marked (G, X)-structures: $(f, M) \longmapsto (f \circ \eta, M)$.
- *Mapping class group*

$$\mathrm{Mod}(\Sigma) := \pi_0(\mathrm{Diff}(\Sigma))$$

acts on $\mathfrak{D}_{(G,X)}(\Sigma)$.

Modeling structures on representations of π_1

- *Marked (G,X)-structure* on Σ: diffeomorphism $\Sigma \xrightarrow{f} M$ where M is a (G,X)-manifold.
- Marked (G,X)-structures (f_i, M_i) are *isotopic* $\iff \exists$ isomorphism $M_1 \xrightarrow{\phi} M_2$ with $\phi \circ f_1 \simeq f_2$.
- Define the *deformation space*

$$\mathfrak{D}_{(G,X)}(\Sigma) := \left\{ \text{Marked } (G,X)\text{-structures on } \Sigma \right\} / \text{Isotopy}$$

- $\eta \in \mathrm{Diff}(\Sigma)$ acts on marked (G,X)-structures: $(f, M) \longmapsto (f \circ \eta, M)$.
- *Mapping class group*

$$\mathrm{Mod}(\Sigma) := \pi_0(\mathrm{Diff}(\Sigma))$$

acts on $\mathfrak{D}_{(G,X)}(\Sigma)$.

Modeling structures on representations of π_1

- *Marked (G,X)-structure* on Σ: diffeomorphism $\Sigma \xrightarrow{f} M$ where M is a (G,X)-manifold.
- Marked (G,X)-structures (f_i, M_i) are *isotopic* $\Longleftrightarrow$ $\exists$ isomorphism $M_1 \xrightarrow{\phi} M_2$ with $\phi \circ f_1 \simeq f_2$.
- Define the *deformation space*

$$\mathfrak{D}_{(G,X)}(\Sigma) := \left\{ \text{Marked } (G,X)\text{-structures on } \Sigma \right\} / \text{Isotopy}$$

- $\eta \in \mathrm{Diff}(\Sigma)$ acts on marked (G,X)-structures: $(f, M) \longmapsto (f \circ \eta, M)$.
- *Mapping class group*

$$\mathrm{Mod}(\Sigma) := \pi_0(\mathrm{Diff}(\Sigma))$$

acts on $\mathfrak{D}_{(G,X)}(\Sigma)$.

Modeling structures on representations of π_1

- *Marked* (G, X)*-structure* on Σ: diffeomorphism $\Sigma \xrightarrow{f} M$ where M is a (G, X)-manifold.
- Marked (G, X)-structures (f_i, M_i) are *isotopic* $\iff \exists$ isomorphism $M_1 \xrightarrow{\phi} M_2$ with $\phi \circ f_1 \simeq f_2$.
- Define the *deformation space*

$$\mathfrak{D}_{(G,X)}(\Sigma) := \left\{ \text{Marked } (G, X)\text{-structures on } \Sigma \right\} / \text{Isotopy}$$

- $\eta \in \text{Diff}(\Sigma)$ acts on marked (G, X)-structures: $(f, M) \longmapsto (f \circ \eta, M)$.
- *Mapping class group*

$$\text{Mod}(\Sigma) := \pi_0\big(\text{Diff}(\Sigma)\big)$$

acts on $\mathfrak{D}_{(G,X)}(\Sigma)$.

Representation varieties

- Let $\pi = \langle X_1, \dots, X_n \rangle$ be finitely generated and $G \subset \mathrm{GL}(N, \mathbb{R})$ a linear algebraic group.
- The set $\mathrm{Hom}(\pi, G)$ of homomorphisms

$$\pi \longrightarrow G$$

  enjoys the natural structure of an *affine algebraic variety.*
- Evaluation on the generators

$$\begin{aligned} \mathrm{Hom}(\pi, G) &\longrightarrow G^n \\ \rho &\longmapsto \big(\rho(X_1), \dots, \rho(X_n)\big) \end{aligned}$$

  embeds $\mathrm{Hom}(\pi, G)$ onto an *algebraic subset* of $\mathrm{GL}(N, \mathbb{R})^N$.
- Structure is $\{X_1, \dots, X_n\}$-independent.
- Classical topology (and Zariski topology).

Representation varieties

- Let $\pi = \langle X_1, \dots, X_n \rangle$ be finitely generated and $G \subset \mathrm{GL}(N, \mathbb{R})$ a linear algebraic group.
- The set $\mathrm{Hom}(\pi, G)$ of homomorphisms

$$\pi \longrightarrow G$$

  enjoys the natural structure of an *affine algebraic variety.*
- Evaluation on the generators

$$\begin{aligned} \mathrm{Hom}(\pi, G) &\longrightarrow G^n \\ \rho &\longmapsto \big(\rho(X_1), \dots, \rho(X_n)\big) \end{aligned}$$

  embeds $\mathrm{Hom}(\pi, G)$ onto an *algebraic subset* of $\mathrm{GL}(N, \mathbb{R})^N$.
- Structure is $\{X_1, \dots, X_n\}$-independent.
- Classical topology (and Zariski topology).

Representation varieties

- Let $\pi = \langle X_1, \ldots, X_n \rangle$ be finitely generated and $G \subset \mathrm{GL}(N, \mathbb{R})$ a linear algebraic group.
- The set $\mathrm{Hom}(\pi, G)$ of homomorphisms

$$\pi \longrightarrow G$$

enjoys the natural structure of an *affine algebraic variety.*

- Evaluation on the generators

$$\begin{aligned} \mathrm{Hom}(\pi, G) &\longrightarrow G^n \\ \rho &\longmapsto \big(\rho(X_1), \ldots, \rho(X_n)\big) \end{aligned}$$

embeds $\mathrm{Hom}(\pi, G)$ onto an *algebraic subset* of $\mathrm{GL}(N, \mathbb{R})^N$.

- Structure is $\{X_1, \ldots, X_n\}$-independent.
- Classical topology (and Zariski topology).

Representation varieties

- Let $\pi = \langle X_1, \ldots, X_n \rangle$ be finitely generated and $G \subset \mathrm{GL}(N, \mathbb{R})$ a linear algebraic group.
- The set $\mathrm{Hom}(\pi, G)$ of homomorphisms

$$\pi \longrightarrow G$$

enjoys the natural structure of an *affine algebraic variety.*

- Evaluation on the generators

$$\begin{aligned} \mathrm{Hom}(\pi, G) &\longrightarrow G^n \\ \rho &\longmapsto \big(\rho(X_1), \ldots, \rho(X_n)\big) \end{aligned}$$

embeds $\mathrm{Hom}(\pi, G)$ onto an *algebraic subset* of $\mathrm{GL}(N, \mathbb{R})^N$.

- Structure is $\{X_1, \ldots, X_n\}$-independent.
- Classical topology (and Zariski topology).

Representation varieties

- Let $\pi = \langle X_1, \ldots, X_n \rangle$ be finitely generated and $G \subset \mathrm{GL}(N, \mathbb{R})$ a linear algebraic group.
- The set $\mathrm{Hom}(\pi, G)$ of homomorphisms

$$\pi \longrightarrow G$$

  enjoys the natural structure of an *affine algebraic variety.*
- Evaluation on the generators

$$\begin{aligned} \mathrm{Hom}(\pi, G) &\longrightarrow G^n \\ \rho &\longmapsto \big(\rho(X_1), \ldots, \rho(X_n)\big) \end{aligned}$$

  embeds $\mathrm{Hom}(\pi, G)$ onto an *algebraic subset* of $\mathrm{GL}(N, \mathbb{R})^N$.
- Structure is $\{X_1, \ldots, X_n\}$-independent.
- Classical topology (and Zariski topology).

Representation varieties

- Let $\pi = \langle X_1, \ldots, X_n \rangle$ be finitely generated and $G \subset \mathrm{GL}(N, \mathbb{R})$ a linear algebraic group.
- The set $\mathrm{Hom}(\pi, G)$ of homomorphisms

$$\pi \longrightarrow G$$

enjoys the natural structure of an *affine algebraic variety.*

- Evaluation on the generators

$$\begin{aligned} \mathrm{Hom}(\pi, G) &\longrightarrow G^n \\ \rho &\longmapsto \big(\rho(X_1), \ldots, \rho(X_n)\big) \end{aligned}$$

embeds $\mathrm{Hom}(\pi, G)$ onto an *algebraic subset* of $\mathrm{GL}(N, \mathbb{R})^N$.

- Structure is $\{X_1, \ldots, X_n\}$-independent.
- Classical topology (and Zariski topology).

Natural symmetries

- $\mathrm{Hom}(\pi, G)$ admits an action of $\mathrm{Aut}(\pi) \times \mathrm{Aut}(G)$:

$$\pi \xrightarrow{\phi^{-1}} \pi \xrightarrow{\rho} G \xrightarrow{\alpha} G$$

  where $(\phi, \alpha) \in \mathrm{Aut}(\pi) \times \mathrm{Aut}(G)$, $\rho \in \mathrm{Hom}(\pi, G)$.
- Preserves the algebraic structure.
- The quotient

$$\mathrm{Hom}(\pi, G)/G \; := \; \mathrm{Hom}(\pi, G)/\big(\{1\} \times \mathrm{Inn}(G)\big)$$

  under the subgroup

$$\{1\} \times \mathrm{Inn}(G) \subset \mathrm{Aut}(\pi) \times \mathrm{Aut}(G)$$

  is the space of equivalence classes of *flat connections* on G-bundles over any space with fundamental group π.
- Inherits an action of

$$\mathrm{Out}(\pi) := \mathrm{Aut}(\pi)/\mathrm{Inn}(\pi).$$

Natural symmetries

- $\mathrm{Hom}(\pi, G)$ admits an action of $\mathrm{Aut}(\pi) \times \mathrm{Aut}(G)$:

$$\pi \xrightarrow{\phi^{-1}} \pi \xrightarrow{\rho} G \xrightarrow{\alpha} G$$

  where $(\phi, \alpha) \in \mathrm{Aut}(\pi) \times \mathrm{Aut}(G)$, $\rho \in \mathrm{Hom}(\pi, G)$.
- Preserves the algebraic structure.
- The quotient

$$\mathrm{Hom}(\pi, G)/G \ := \ \mathrm{Hom}(\pi, G)/(\{1\} \times \mathrm{Inn}(G))$$

  under the subgroup

$$\{1\} \times \mathrm{Inn}(G) \subset \mathrm{Aut}(\pi) \times \mathrm{Aut}(G)$$

  is the space of equivalence classes of *flat connections* on G-bundles over any space with fundamental group π.
- Inherits an action of

$$\mathrm{Out}(\pi) := \mathrm{Aut}(\pi)/\mathrm{Inn}(\pi).$$

Natural symmetries

- $\mathrm{Hom}(\pi, G)$ admits an action of $\mathrm{Aut}(\pi) \times \mathrm{Aut}(G)$:

$$\pi \xrightarrow{\phi^{-1}} \pi \xrightarrow{\rho} G \xrightarrow{\alpha} G$$

  where $(\phi, \alpha) \in \mathrm{Aut}(\pi) \times \mathrm{Aut}(G)$, $\rho \in \mathrm{Hom}(\pi, G)$.
- Preserves the algebraic structure.
- The quotient

$$\mathrm{Hom}(\pi, G)/G \; := \; \mathrm{Hom}(\pi, G)/(\{1\} \times \mathrm{Inn}(G))$$

  under the subgroup

$$\{1\} \times \mathrm{Inn}(G) \subset \mathrm{Aut}(\pi) \times \mathrm{Aut}(G)$$

  is the space of equivalence classes of *flat connections* on G-bundles over any space with fundamental group π.
- Inherits an action of

$$\mathrm{Out}(\pi) := \mathrm{Aut}(\pi)/\mathrm{Inn}(\pi).$$

Natural symmetries

- $\mathrm{Hom}(\pi, G)$ admits an action of $\mathrm{Aut}(\pi) \times \mathrm{Aut}(G)$:

$$\pi \xrightarrow{\phi^{-1}} \pi \xrightarrow{\rho} G \xrightarrow{\alpha} G$$

  where $(\phi, \alpha) \in \mathrm{Aut}(\pi) \times \mathrm{Aut}(G)$, $\rho \in \mathrm{Hom}(\pi, G)$.
- Preserves the algebraic structure.
- The quotient

$$\mathrm{Hom}(\pi, G)/G \ := \ \mathrm{Hom}(\pi, G)/\big(\{1\} \times \mathrm{Inn}(G)\big)$$

  under the subgroup

$$\{1\} \times \mathrm{Inn}(G) \subset \mathrm{Aut}(\pi) \times \mathrm{Aut}(G)$$

  is the space of equivalence classes of *flat connections* on G-bundles over any space with fundamental group π.
- Inherits an action of

$$\mathrm{Out}(\pi) := \mathrm{Aut}(\pi)/\mathrm{Inn}(\pi).$$

Natural symmetries

- $\mathsf{Hom}(\pi, G)$ admits an action of $\mathsf{Aut}(\pi) \times \mathsf{Aut}(G)$:

$$\pi \xrightarrow{\phi^{-1}} \pi \xrightarrow{\rho} G \xrightarrow{\alpha} G$$

  where $(\phi, \alpha) \in \mathsf{Aut}(\pi) \times \mathsf{Aut}(G)$, $\rho \in \mathsf{Hom}(\pi, G)$.
- Preserves the algebraic structure.
- The quotient

$$\mathsf{Hom}(\pi, G)/G \ := \ \mathsf{Hom}(\pi, G)/\big(\{1\} \times \mathsf{Inn}(G)\big)$$

  under the subgroup

$$\{1\} \times \mathsf{Inn}(G) \subset \mathsf{Aut}(\pi) \times \mathsf{Aut}(G)$$

  is the space of equivalence classes of *flat connections* on G-bundles over any space with fundamental group π.
- Inherits an action of

$$\mathsf{Out}(\pi) := \mathsf{Aut}(\pi)/\mathsf{Inn}(\pi).$$

Holonomy

- A marked structure determines a *developing map* $\tilde{\Sigma} \longrightarrow X$ and a *holonomy representation* $\pi \longrightarrow G$.
- Globalize the coordinate charts and coordinate changes respectively.
- Well-defined up to transformations in G.
- Holonomy defines a mapping

$$\mathfrak{D}_{(G,X)}(\Sigma) \xrightarrow{\text{hol}} \mathrm{Hom}(\pi, G)/G$$

- Equivariant respecting

$$\mathrm{Mod}(\Sigma) \longrightarrow \mathrm{Out}\big(\pi_1(\Sigma)\big)$$

- (Thurston): The mapping hol is a local homeomorphism.
- For quotient structures, hol is an embedding.

Holonomy

- A marked structure determines a *developing map* $\tilde{\Sigma} \longrightarrow X$ and a *holonomy representation* $\pi \longrightarrow G$.
- Globalize the coordinate charts and coordinate changes respectively.
- Well-defined up to transformations in G.
- Holonomy defines a mapping

$$\mathfrak{D}_{(G,X)}(\Sigma) \xrightarrow{\text{hol}} \mathrm{Hom}(\pi, G)/G$$

- Equivariant respecting

$$\mathrm{Mod}(\Sigma) \longrightarrow \mathrm{Out}\big(\pi_1(\Sigma)\big)$$

- (Thurston): The mapping hol is a local homeomorphism.
- For quotient structures, hol is an embedding.

Holonomy

- A marked structure determines a *developing map* $\tilde{\Sigma} \longrightarrow X$ and a *holonomy representation* $\pi \longrightarrow G$.
- Globalize the coordinate charts and coordinate changes respectively.
- Well-defined up to transformations in G.
- Holonomy defines a mapping

$$\mathfrak{D}_{(G,X)}(\Sigma) \xrightarrow{\text{hol}} \text{Hom}(\pi, G)/G$$

- Equivariant respecting

$$\text{Mod}(\Sigma) \longrightarrow \text{Out}\big(\pi_1(\Sigma)\big)$$

- (Thurston): The mapping hol is a local homeomorphism.
- For quotient structures, hol is an embedding.

Holonomy

- A marked structure determines a *developing map* $\tilde{\Sigma} \longrightarrow X$ and a *holonomy representation* $\pi \longrightarrow G$.
- Globalize the coordinate charts and coordinate changes respectively.
- Well-defined up to transformations in G.
- Holonomy defines a mapping

$$\mathfrak{D}_{(G,X)}(\Sigma) \xrightarrow{\text{hol}} \mathrm{Hom}(\pi, G)/G$$

- Equivariant respecting

$$\mathrm{Mod}(\Sigma) \longrightarrow \mathrm{Out}\big(\pi_1(\Sigma)\big)$$

- (Thurston): The mapping hol is a local homeomorphism.
- For quotient structures, hol is an embedding.

Holonomy

- A marked structure determines a *developing map* $\tilde{\Sigma} \longrightarrow X$ and a *holonomy representation* $\pi \longrightarrow G$.
- Globalize the coordinate charts and coordinate changes respectively.
- Well-defined up to transformations in G.
- Holonomy defines a mapping

$$\mathfrak{D}_{(G,X)}(\Sigma) \xrightarrow{\text{hol}} \text{Hom}(\pi, G)/G$$

- Equivariant respecting

$$\text{Mod}(\Sigma) \longrightarrow \text{Out}\big(\pi_1(\Sigma)\big)$$

- (Thurston): The mapping hol is a local homeomorphism.
- For quotient structures, hol is an embedding.

Holonomy

- A marked structure determines a *developing map* $\tilde{\Sigma} \longrightarrow X$ and a *holonomy representation* $\pi \longrightarrow G$.
- Globalize the coordinate charts and coordinate changes respectively.
- Well-defined up to transformations in G.
- Holonomy defines a mapping

$$\mathfrak{D}_{(G,X)}(\Sigma) \xrightarrow{\text{hol}} \mathrm{Hom}(\pi, G)/G$$

- Equivariant respecting

$$\mathrm{Mod}(\Sigma) \longrightarrow \mathrm{Out}\big(\pi_1(\Sigma)\big)$$

- (Thurston): The mapping hol is a local homeomorphism.
- For quotient structures, hol is an embedding.

Holonomy

- A marked structure determines a *developing map* $\tilde{\Sigma} \longrightarrow X$ and a *holonomy representation* $\pi \longrightarrow G$.
- Globalize the coordinate charts and coordinate changes respectively.
- Well-defined up to transformations in G.
- Holonomy defines a mapping

$$\mathfrak{D}_{(G,X)}(\Sigma) \xrightarrow{\text{hol}} \mathrm{Hom}(\pi, G)/G$$

- Equivariant respecting

$$\mathrm{Mod}(\Sigma) \longrightarrow \mathrm{Out}\big(\pi_1(\Sigma)\big)$$

- (Thurston): The mapping hol is a local homeomorphism.
- For quotient structures, hol is an embedding.

Holonomy

- A marked structure determines a *developing map* $\tilde{\Sigma} \longrightarrow X$ and a *holonomy representation* $\pi \longrightarrow G$.
- Globalize the coordinate charts and coordinate changes respectively.
- Well-defined up to transformations in G.
- Holonomy defines a mapping

$$\mathfrak{D}_{(G,X)}(\Sigma) \xrightarrow{\text{hol}} \mathrm{Hom}(\pi, G)/G$$

- Equivariant respecting

$$\mathrm{Mod}(\Sigma) \longrightarrow \mathrm{Out}\big(\pi_1(\Sigma)\big)$$

- (Thurston): The mapping hol is a local homeomorphism.
- For quotient structures, hol is an embedding.

Examples: Hyperbolic structures

- *Hyperbolic geometry:* When $X = \mathsf{H}^2$ and $G = \mathsf{Isom}(\mathsf{H}^2)$, the deformation space $\mathfrak{D}_{(G,X)}(\Sigma)$ identifies with the Fricke-Teichmüller space $\mathfrak{F}(\Sigma)$ of Σ.
- Identifies with Teichmüller space (marked *conformal* structures via *Uniformization Theorem*.
- hol embeds $\mathfrak{F}(\Sigma)$ as a *connected component* of $\mathsf{Hom}(\pi, G)/G$.
- $\mathfrak{F}(\Sigma) \approx \mathbb{R}^{6g-6}$ and $\mathsf{Mod}(\Sigma)$ acts properly discretely.

Examples: Hyperbolic structures

- *Hyperbolic geometry:* When $X = \mathsf{H}^2$ and $G = \mathsf{Isom}(\mathsf{H}^2)$, the deformation space $\mathfrak{D}_{(G,X)}(\Sigma)$ identifies with the Fricke-Teichmüller space $\mathfrak{F}(\Sigma)$ of Σ.
- Identifies with Teichmüller space (marked *conformal* structures via *Uniformization Theorem.*
- hol embeds $\mathfrak{F}(\Sigma)$ as a *connected component* of $\mathsf{Hom}(\pi, G)/G$).
- $\mathfrak{F}(\Sigma) \approx \mathbb{R}^{6g-6}$ and $\mathsf{Mod}(\Sigma)$ acts properly discretely.

Examples: Hyperbolic structures

- *Hyperbolic geometry:* When $X = \mathsf{H}^2$ and $G = \mathsf{Isom}(\mathsf{H}^2)$, the deformation space $\mathfrak{D}_{(G,X)}(\Sigma)$ identifies with the Fricke-Teichmüller space $\mathfrak{F}(\Sigma)$ of Σ.
- Identifies with Teichmüller space (marked *conformal* structures via *Uniformization Theorem.*
- hol embeds $\mathfrak{F}(\Sigma)$ as a *connected component* of $\mathsf{Hom}(\pi, G)/G$.
- $\mathfrak{F}(\Sigma) \approx \mathbb{R}^{6g-6}$ and $\mathsf{Mod}(\Sigma)$ acts properly discretely.

Examples: Hyperbolic structures

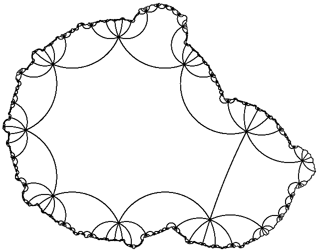
- *Hyperbolic geometry:* When $X = \mathrm{H}^2$ and $G = \mathrm{Isom}(\mathrm{H}^2)$, the deformation space $\mathfrak{D}_{(G,X)}(\Sigma)$ identifies with the Fricke-Teichmüller space $\mathfrak{F}(\Sigma)$ of Σ.
- Identifies with Teichmüller space (marked *conformal* structures via *Uniformization Theorem.*
- hol embeds $\mathfrak{F}(\Sigma)$ as a *connected component* of $\mathrm{Hom}(\pi, G)/G$.
- $\mathfrak{F}(\Sigma) \approx \mathbb{R}^{6g-6}$ and $\mathrm{Mod}(\Sigma)$ acts properly discretely.

Examples: Hyperbolic structures

- *Hyperbolic geometry:* When $X = \mathsf{H}^2$ and $G = \mathsf{Isom}(\mathsf{H}^2)$, the deformation space $\mathfrak{D}_{(G,X)}(\Sigma)$ identifies with the Fricke-Teichmüller space $\mathfrak{F}(\Sigma)$ of Σ.
- Identifies with Teichmüller space (marked *conformal* structures via *Uniformization Theorem.*
- hol embeds $\mathfrak{F}(\Sigma)$ as a *connected component* of $\mathsf{Hom}(\pi, G)/G$.
- $\mathfrak{F}(\Sigma) \approx \mathbb{R}^{6g-6}$ and $\mathsf{Mod}(\Sigma)$ acts properly discretely.

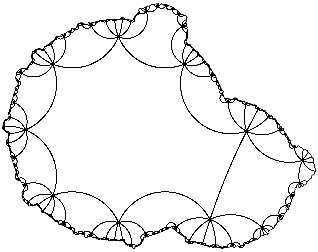
Example: $\mathbb{CP}^1$-structures

- When $X = \mathbb{CP}^1$ and $G = \mathrm{PGL}(2,\mathbb{C})$, Poincaré identified $\mathfrak{D}_{(G,X)}(\Sigma)$ with an affine bundle over $\mathfrak{F}(\Sigma)$ whose fiber over a Riemann surface R is the vector space $H^0(R, K^2)$ of *holomorphic quadratic differentials.*
- Thus $\mathfrak{D}_{(G,X)}(\Sigma) \approx \mathbb{R}^{12g-12}$ and $\mathrm{Mod}(\Sigma)$ acts properly discretely.
- $\mathfrak{D}_{(G,X)}(\Sigma)$ contains the space of *quasi-Fuchsian* representations.



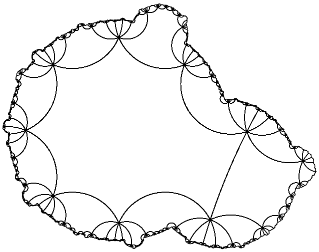
Example: $\mathbb{CP}^1$-structures

- When $X = \mathbb{CP}^1$ and $G = \mathrm{PGL}(2,\mathbb{C})$, Poincaré identified $\mathfrak{D}_{(G,X)}(\Sigma)$ with an affine bundle over $\mathfrak{F}(\Sigma)$ whose fiber over a Riemann surface R is the vector space $H^0(R, K^2)$ of *holomorphic quadratic differentials.*
- Thus $\mathfrak{D}_{(G,X)}(\Sigma) \approx \mathbb{R}^{12g-12}$ and $\mathrm{Mod}(\Sigma)$ acts properly discretely.
- $\mathfrak{D}_{(G,X)}(\Sigma)$ contains the space of *quasi-Fuchsian* representations.



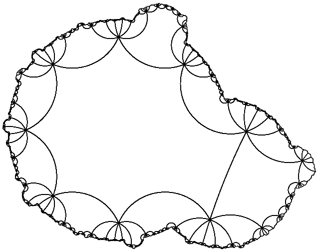
Example: $\mathbb{CP}^1$-structures

- When $X = \mathbb{CP}^1$ and $G = \mathrm{PGL}(2, \mathbb{C})$, Poincaré identified $\mathfrak{D}_{(G,X)}(\Sigma)$ with an affine bundle over $\mathfrak{F}(\Sigma)$ whose fiber over a Riemann surface R is the vector space $H^0(R, K^2)$ of *holomorphic quadratic differentials.*
- Thus $\mathfrak{D}_{(G,X)}(\Sigma) \approx \mathbb{R}^{12g-12}$ and $\mathrm{Mod}(\Sigma)$ acts properly discretely.
- $\mathfrak{D}_{(G,X)}(\Sigma)$ contains the space of *quasi-Fuchsian* representations.



Example: $\mathbb{CP}^1$-structures

- When $X = \mathbb{CP}^1$ and $G = \mathrm{PGL}(2, \mathbb{C})$, Poincaré identified $\mathfrak{D}_{(G,X)}(\Sigma)$ with an affine bundle over $\mathfrak{F}(\Sigma)$ whose fiber over a Riemann surface R is the vector space $H^0(R, K^2)$ of *holomorphic quadratic differentials.*
- Thus $\mathfrak{D}_{(G,X)}(\Sigma) \approx \mathbb{R}^{12g-12}$ and $\mathrm{Mod}(\Sigma)$ acts properly discretely.
- $\mathfrak{D}_{(G,X)}(\Sigma)$ contains the space of *quasi-Fuchsian* representations.



Example: $\mathbb{RP}^2$-structures

- When $X = \mathbb{RP}^2$ and $G = \mathrm{PGL}(3, \mathbb{R})$, the deformation space $\mathfrak{D}_{(G,X)}(\Sigma)$ is a vector bundle over $\mathfrak{F}(\Sigma)$ whose fiber over a Riemann surface R is the vector space $H^0(R, K^3)$ of *holomorphic cubic differentials* (Labourie, Loftin)
- For any $\mathbb{R}$-split semisimple G, Hitchin found a contractible component containing $\mathfrak{F}(\Sigma)$.
- More recently, Labourie showed all representations in this component are discrete embeddings and that $\mathrm{Mod}(\Sigma)$ acts properly discretely.
- Markedly different from $\mathbb{CP}^1$-structures, where the holonomy representations may not be discrete embeddings and hol is very complicated.

Example: $\mathbb{RP}^2$-structures

- When $X = \mathbb{RP}^2$ and $G = \mathrm{PGL}(3, \mathbb{R})$, the deformation space $\mathfrak{D}_{(G,X)}(\Sigma)$ is a vector bundle over $\mathfrak{F}(\Sigma)$ whose fiber over a Riemann surface R is the vector space $H^0(R, K^3)$ of *holomorphic cubic differentials* (Labourie, Loftin)
- For any $\mathbb{R}$-split semisimple G, Hitchin found a contractible component containing $\mathfrak{F}(\Sigma)$.
- More recently, Labourie showed all representations in this component are discrete embeddings and that $\mathrm{Mod}(\Sigma)$ acts properly discretely.
- Markedly different from $\mathbb{CP}^1$-structures, where the holonomy representations may not be discrete embeddings and hol is very complicated.

Example: $\mathbb{RP}^2$-structures

- When $X = \mathbb{RP}^2$ and $G = \mathrm{PGL}(3, \mathbb{R})$, the deformation space $\mathfrak{D}_{(G,X)}(\Sigma)$ is a vector bundle over $\mathfrak{F}(\Sigma)$ whose fiber over a Riemann surface R is the vector space $H^0(R, K^3)$ of *holomorphic cubic differentials* (Labourie, Loftin)
- For any $\mathbb{R}$-split semisimple G, Hitchin found a contractible component containing $\mathfrak{F}(\Sigma)$.
- More recently, Labourie showed all representations in this component are discrete embeddings and that $\mathrm{Mod}(\Sigma)$ acts properly discretely.
- Markedly different from $\mathbb{CP}^1$-structures, where the holonomy representations may not be discrete embeddings and hol is very complicated.

Example: $\mathbb{RP}^2$-structures

- When $X = \mathbb{RP}^2$ and $G = \mathrm{PGL}(3,\mathbb{R})$, the deformation space $\mathfrak{D}_{(G,X)}(\Sigma)$ is a vector bundle over $\mathfrak{F}(\Sigma)$ whose fiber over a Riemann surface R is the vector space $H^0(R, K^3)$ of *holomorphic cubic differentials* (Labourie, Loftin)
- For any $\mathbb{R}$-split semisimple G, Hitchin found a contractible component containing $\mathfrak{F}(\Sigma)$.
- More recently, Labourie showed all representations in this component are discrete embeddings and that $\mathrm{Mod}(\Sigma)$ acts properly discretely.
- Markedly different from $\mathbb{CP}^1$-structures, where the holonomy representations may not be discrete embeddings and hol is very complicated.

Example: $\mathbb{RP}^2$-structures

- When $X = \mathbb{RP}^2$ and $G = \mathrm{PGL}(3, \mathbb{R})$, the deformation space $\mathfrak{D}_{(G,X)}(\Sigma)$ is a vector bundle over $\mathfrak{F}(\Sigma)$ whose fiber over a Riemann surface R is the vector space $H^0(R, K^3)$ of *holomorphic cubic differentials* (Labourie, Loftin)
- For any $\mathbb{R}$-split semisimple G, Hitchin found a contractible component containing $\mathfrak{F}(\Sigma)$.
- More recently, Labourie showed all representations in this component are discrete embeddings and that $\mathrm{Mod}(\Sigma)$ acts properly discretely.
- Markedly different from $\mathbb{CP}^1$-structures, where the holonomy representations may not be discrete embeddings and hol is very complicated.

Character variety

- Even though G acts algebraically on $\mathrm{Hom}(\pi, G)$, the quotient $\mathrm{Hom}(\pi, G)/G$ is not algebraic.
- The GIT quotient $\mathrm{Hom}(\pi, G)//G$ is an $\mathrm{Out}(\pi)$-invariant affine algebraic variety.
- Coordinate ring is the invariant subring

$$\mathbb{C}[\mathrm{Hom}(\pi, G)//G] = \mathbb{C}[\mathrm{Hom}(\pi, G)]^G \subset \mathbb{C}[\mathrm{Hom}(\pi, G)].$$

- Examples are functions f_α, associated to:
  - A conjugacy class $[\alpha]$, where $\alpha \in \pi$;
  - An $\mathrm{Inn}(G)$-invariant function $G \xrightarrow{f} \mathbb{R}$.

Character variety

- Even though G acts algebraically on $\mathrm{Hom}(\pi, G)$, the quotient $\mathrm{Hom}(\pi, G)/G)$ is not algebraic.
- The GIT quotient $\mathrm{Hom}(\pi, G)//G$ is an $\mathrm{Out}(\pi)$-invariant affine algebraic variety.
- Coordinate ring is the invariant subring

$$\mathbb{C}[\mathrm{Hom}(\pi, G)//G] = \mathbb{C}[\mathrm{Hom}(\pi, G)]^G \subset \mathbb{C}[\mathrm{Hom}(\pi, G)].$$

- Examples are functions f_α, associated to:
  - A conjugacy class $[\alpha]$, where $\alpha \in \pi$;
  - An $\mathrm{Inn}(G)$-invariant function $G \xrightarrow{f} \mathbb{R}$.

Character variety

- Even though G acts algebraically on $\mathrm{Hom}(\pi, G)$, the quotient $\mathrm{Hom}(\pi, G)/G)$ is not algebraic.
- The GIT quotient $\mathrm{Hom}(\pi, G)//G$ is an $\mathrm{Out}(\pi)$-invariant affine algebraic variety.
- Coordinate ring is the invariant subring

$$\mathbb{C}[\mathrm{Hom}(\pi, G)//G] = \mathbb{C}[\mathrm{Hom}(\pi, G)]^G \subset \mathbb{C}[\mathrm{Hom}(\pi, G)].$$

- Examples are functions f_α, associated to:
  - A conjugacy class $[\alpha]$, where $\alpha \in \pi$;
  - An $\mathrm{Inn}(G)$-invariant function $G \xrightarrow{f} \mathbb{R}$.

Character variety

- Even though G acts algebraically on $\mathsf{Hom}(\pi, G)$, the quotient $\mathsf{Hom}(\pi, G)/G)$ is not algebraic.
- The GIT quotient $\mathsf{Hom}(\pi, G)//G$ is an $\mathsf{Out}(\pi)$-invariant affine algebraic variety.
- Coordinate ring is the invariant subring

$$\mathbb{C}[\mathsf{Hom}(\pi, G)//G] = \mathbb{C}[\mathsf{Hom}(\pi, G)]^G \subset \mathbb{C}[\mathsf{Hom}(\pi, G)].$$

- Examples are functions f_α, associated to:
  - A conjugacy class $[\alpha]$, where $\alpha \in \pi$;
  - An $\mathsf{Inn}(G)$-invariant function $G \xrightarrow{f} \mathbb{R}$.

Character variety

- Even though G acts algebraically on $\text{Hom}(\pi, G)$, the quotient $\text{Hom}(\pi, G)/G)$ is not algebraic.
- The GIT quotient $\text{Hom}(\pi, G)//G$ is an $\text{Out}(\pi)$-invariant affine algebraic variety.
- Coordinate ring is the invariant subring

$$\mathbb{C}[\text{Hom}(\pi, G)//G] = \mathbb{C}[\text{Hom}(\pi, G)]^G \subset \mathbb{C}[\text{Hom}(\pi, G)].$$

- Examples are functions f_α, associated to:
  - A conjugacy class $[\alpha]$, where $\alpha \in \pi$;
  - An $\text{Inn}(G)$-invariant function $G \xrightarrow{f} \mathbb{R}$.

Character variety

- Even though G acts algebraically on $\mathrm{Hom}(\pi, G)$, the quotient $\mathrm{Hom}(\pi, G)/G)$ is not algebraic.
- The GIT quotient $\mathrm{Hom}(\pi, G)//G$ is an $\mathrm{Out}(\pi)$-invariant affine algebraic variety.
- Coordinate ring is the invariant subring

$$\mathbb{C}[\mathrm{Hom}(\pi, G)//G] = \mathbb{C}[\mathrm{Hom}(\pi, G)]^G \subset \mathbb{C}[\mathrm{Hom}(\pi, G)].$$

- Examples are functions f_α, associated to:
  - A conjugacy class $[\alpha]$, where $\alpha \in \pi$;
  - An $\mathrm{Inn}(G)$-invariant function $G \xrightarrow{f} \mathbb{R}$.

Character variety

- Even though G acts algebraically on $\mathrm{Hom}(\pi, G)$, the quotient $\mathrm{Hom}(\pi, G)/G)$ is not algebraic.
- The GIT quotient $\mathrm{Hom}(\pi, G)//G$ is an $\mathrm{Out}(\pi)$-invariant affine algebraic variety.
- Coordinate ring is the invariant subring

$$\mathbb{C}[\mathrm{Hom}(\pi, G)//G] = \mathbb{C}[\mathrm{Hom}(\pi, G)]^G \subset \mathbb{C}[\mathrm{Hom}(\pi, G)].$$

- Examples are functions f_α, associated to:
  - A conjugacy class $[\alpha]$, where $\alpha \in \pi$;
  - An $\mathrm{Inn}(G)$-invariant function $G \xrightarrow{f} \mathbb{R}$.

Character functions f_α on representation varieties

- Conjugacy class of $\alpha \in \pi$ corresponds to free homotopy class of closed oriented loop $\alpha \subset \Sigma$.
- Invariant function $G \xrightarrow{f} \mathbb{R} \Longrightarrow$ *Function* f_α on $\mathrm{Hom}(\pi, G)/G$:

$$\begin{aligned} \mathrm{Hom}(\pi, G)/G &\xrightarrow{f_\alpha} \mathbb{R} \\ [\rho] &\longmapsto f\big(\rho(\alpha)\big) \end{aligned}$$

- Example: *Trace* $\mathrm{GL}(n, \mathbb{R}) \xrightarrow{\mathrm{tr}} \mathbb{R}$
- Another example: *Displacement length*

$$\ell(A) := \min_{x \in \mathsf{H}^2} d\big(x, A(x)\big)$$

$$\mathrm{tr}(A) = \pm 2 \cosh\big(\ell(A)/2\big) \text{ if } A \text{ hyperbolic}$$

- On $\mathfrak{F}(\Sigma)$, ℓ_α associates to a marked hyperbolic surface $\Sigma \approx M$ length of the unique closed geodesic homotopic to α in M.

Character functions f_α on representation varieties

- Conjugacy class of $\alpha \in \pi$ corresponds to free homotopy class of closed oriented loop $\alpha \subset \Sigma$.
- Invariant function $G \xrightarrow{f} \mathbb{R} \Longrightarrow$ *Function* f_α on $\mathrm{Hom}(\pi, G)/G$:

$$\mathrm{Hom}(\pi, G)/G \xrightarrow{f_\alpha} \mathbb{R}$$
$$[\rho] \longmapsto f\big(\rho(\alpha)\big)$$

- Example: *Trace* $\mathrm{GL}(n,\mathbb{R}) \xrightarrow{\mathrm{tr}} \mathbb{R}$
- Another example: *Displacement length*

$$\ell(A) := \min_{x \in \mathrm{H}^2} d\big(x, A(x)\big)$$

$$\mathrm{tr}(A) = \pm 2\cosh\big(\ell(A)/2\big) \text{ if } A \text{ hyperbolic}$$

- On $\mathfrak{F}(\Sigma)$, ℓ_α associates to a marked hyperbolic surface $\Sigma \approx M$ length of the unique closed geodesic homotopic to α in M.

Character functions f_α on representation varieties

- Conjugacy class of $\alpha \in \pi$ corresponds to free homotopy class of closed oriented loop $\alpha \subset \Sigma$.
- Invariant function $G \xrightarrow{f} \mathbb{R} \Longrightarrow$ *Function* f_α on $\mathrm{Hom}(\pi, G)/G$:

$$\mathrm{Hom}(\pi, G)/G \xrightarrow{f_\alpha} \mathbb{R}$$
$$[\rho] \longmapsto f\big(\rho(\alpha)\big)$$

- Example: *Trace* $\mathrm{GL}(n, \mathbb{R}) \xrightarrow{\mathrm{tr}} \mathbb{R}$
- Another example: *Displacement length*

$$\ell(A) := \min_{x \in \mathrm{H}^2} d\big(x, A(x)\big)$$

$$\mathrm{tr}(A) = \pm 2 \cosh\big(\ell(A)/2\big) \text{ if } A \text{ hyperbolic}$$

- On $\mathfrak{F}(\Sigma)$, ℓ_α associates to a marked hyperbolic surface $\Sigma \approx M$ length of the unique closed geodesic homotopic to α in M.

Character functions f_α on representation varieties

- Conjugacy class of $\alpha \in \pi$ corresponds to free homotopy class of closed oriented loop $\alpha \subset \Sigma$.
- Invariant function $G \xrightarrow{f} \mathbb{R} \Longrightarrow$ *Function* f_α on $\mathrm{Hom}(\pi, G)/G$:

$$\mathrm{Hom}(\pi, G)/G \xrightarrow{f_\alpha} \mathbb{R}$$
$$[\rho] \longmapsto f\big(\rho(\alpha)\big)$$

- Example: *Trace* $\mathrm{GL}(n,\mathbb{R}) \xrightarrow{\mathrm{tr}} \mathbb{R}$
- Another example: *Displacement length*

$$\ell(A) := \min_{x \in \mathrm{H}^2} d\big(x, A(x)\big)$$

$$\mathrm{tr}(A) = \pm 2\cosh\big(\ell(A)/2\big) \text{ if } A \text{ hyperbolic}$$

- On $\mathfrak{F}(\Sigma)$, ℓ_α associates to a marked hyperbolic surface $\Sigma \approx M$ length of the unique closed geodesic homotopic to α in M.

Character functions f_α on representation varieties

- Conjugacy class of $\alpha \in \pi$ corresponds to free homotopy class of closed oriented loop $\alpha \subset \Sigma$.
- Invariant function $G \xrightarrow{f} \mathbb{R} \Longrightarrow$ *Function* f_α on $\mathrm{Hom}(\pi, G)/G$:

$$\begin{aligned} \mathrm{Hom}(\pi, G)/G &\xrightarrow{f_\alpha} \mathbb{R} \\ [\rho] &\longmapsto f\big(\rho(\alpha)\big) \end{aligned}$$

- Example: *Trace* $\mathrm{GL}(n, \mathbb{R}) \xrightarrow{\mathrm{tr}} \mathbb{R}$
- Another example: *Displacement length*

$$\ell(A) := \min_{x \in \mathrm{H}^2} d\big(x, A(x)\big)$$

$$\mathrm{tr}(A) = \pm 2 \cosh\big(\ell(A)/2\big) \text{ if } A \text{ hyperbolic}$$

- On $\mathfrak{F}(\Sigma)$, ℓ_α associates to a marked hyperbolic surface $\Sigma \approx M$ length of the unique closed geodesic homotopic to α in M.

Character functions f_α on representation varieties

- Conjugacy class of $\alpha \in \pi$ corresponds to free homotopy class of closed oriented loop $\alpha \subset \Sigma$.
- Invariant function $G \xrightarrow{f} \mathbb{R} \Longrightarrow$ *Function* f_α on $\mathrm{Hom}(\pi, G)/G$:

$$\begin{aligned} \mathrm{Hom}(\pi, G)/G &\xrightarrow{f_\alpha} \mathbb{R} \\ [\rho] &\longmapsto f\big(\rho(\alpha)\big) \end{aligned}$$

- Example: *Trace* $\mathrm{GL}(n, \mathbb{R}) \xrightarrow{\mathrm{tr}} \mathbb{R}$
- Another example: *Displacement length*

$$\ell(A) := \min_{x \in \mathrm{H}^2} d\big(x, A(x)\big)$$

$$\mathrm{tr}(A) = \pm 2 \cosh\big(\ell(A)/2\big) \text{ if } A \text{ hyperbolic}$$

- On $\mathfrak{F}(\Sigma)$, ℓ_α associates to a marked hyperbolic surface $\Sigma \approx M$ length of the unique closed geodesic homotopic to α in M.

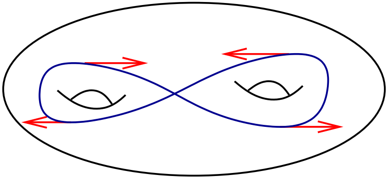
Hamiltonian twist flows on $\mathrm{Hom}(\pi, G)$

- The Hamiltonian vector field $\mathrm{Ham}(f_\alpha)$ associated to f and α assigns to a representation ρ in $\mathrm{Hom}(\pi, G)$ a tangent vector

$$\mathrm{Ham}(f_\alpha)[\rho] \in T_{[\rho]}\mathrm{Hom}(\pi, G)/G = H^1(\Sigma, \mathfrak{g}_{\mathrm{Ad}\rho}).$$

- It is represented by the (Poincaré dual) cycle-with-coefficient supported on α and with coefficient

$$F(\rho(\alpha)) \in \mathfrak{g}_{\mathrm{Ad}\rho}.$$



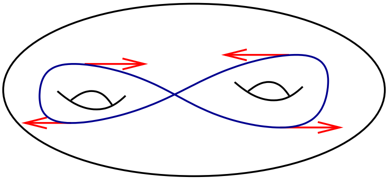
Hamiltonian twist flows on $\mathrm{Hom}(\pi, G)$

- The Hamiltonian vector field $\mathrm{Ham}(f_\alpha)$ associated to f and α assigns to a representation ρ in $\mathrm{Hom}(\pi, G)$ a tangent vector

$$\mathrm{Ham}(f_\alpha)[\rho] \in T_{[\rho]}\mathrm{Hom}(\pi, G)/G = H^1(\Sigma, \mathfrak{g}_{\mathrm{Ad}\rho}).$$

- It is represented by the (Poincaré dual) cycle-with-coefficient supported on α and with coefficient

$$F(\rho(\alpha)) \in \mathfrak{g}_{\mathrm{Ad}\rho}.$$



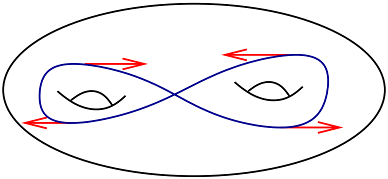
Hamiltonian twist flows on $\mathrm{Hom}(\pi, G)$

- The Hamiltonian vector field $\mathrm{Ham}(f_\alpha)$ associated to f and α assigns to a representation ρ in $\mathrm{Hom}(\pi, G)$ a tangent vector

$$\mathrm{Ham}(f_\alpha)[\rho] \in T_{[\rho]}\mathrm{Hom}(\pi, G)/G = H^1(\Sigma, \mathfrak{g}_{\mathrm{Ad}\rho}).$$

- It is represented by the (Poincaré dual) cycle-with-coefficient supported on α and with coefficient

$$F(\rho(\alpha)) \in \mathfrak{g}_{\mathrm{Ad}\rho}.$$



The one-parameter subgroup associated to an invariant function

- Invariant function

$$G \xrightarrow{f} \mathbb{R}$$

and $A \in G \implies$ one-parameter subgroup

$$\zeta(t) = \exp\big(tF(A)\big) \in G,$$

where $F(A) \in \mathfrak{g}$.

- Centralizes A:

$$\zeta(t)A\zeta^{-1} = A$$

- $F(A)$ is defined by duality:

$$df(A) \in T_A^*G \cong \mathfrak{g}^* \overset{\mathbb{B}}{\cong} \mathfrak{g}$$

The one-parameter subgroup associated to an invariant function

- Invariant function

$$G \xrightarrow{f} \mathbb{R}$$

and $A \in G \implies$ one-parameter subgroup

$$\zeta(t) = \exp\big(tF(A)\big) \in G,$$

where $F(A) \in \mathfrak{g}$.

- Centralizes A:

$$\zeta(t)A\zeta^{-1} = A$$

- $F(A)$ is defined by duality:

$$df(A) \in T_A^*G \cong \mathfrak{g}^* \overset{\mathbb{B}}{\cong} \mathfrak{g}$$

The one-parameter subgroup associated to an invariant function

- Invariant function

$$G \xrightarrow{f} \mathbb{R}$$

and $A \in G \implies$ one-parameter subgroup

$$\zeta(t) = \exp\big(tF(A)\big) \in G,$$

where $F(A) \in \mathfrak{g}$.

- Centralizes A:

$$\zeta(t)A\zeta^{-1} = A$$

- $F(A)$ is defined by duality:

$$df(A) \in T_A^*G \cong \mathfrak{g}^* \overset{\mathbb{B}}{\cong} \mathfrak{g}$$

The one-parameter subgroup associated to an invariant function

- Invariant function
$$G \xrightarrow{f} \mathbb{R}$$
and $A \in G \implies$ one-parameter subgroup
$$\zeta(t) = \exp\big(tF(A)\big) \in G,$$
where $F(A) \in \mathfrak{g}$.
- Centralizes A:
$$\zeta(t)A\zeta^{-1} = A$$
- $F(A)$ is defined by duality:
$$df(A) \in T_A^*G \cong \mathfrak{g}^* \overset{\mathbb{B}}{\cong} \mathfrak{g}$$

Generalized twist deformations

- When α is a simple closed curve, then a flow Φ_t on $\mathrm{Hom}(\pi, G)$ exists, which covers the (local) flow of the Hamiltonian vector field $\mathrm{Ham}(f_\alpha)$.
- Example: α nonseparating curve A_1 in

$$\pi = \langle A_1, B_1, \dots, A_g, B_g \mid A_1 B_1 A_1^{-1} B_1^{-1} \dots, A_g B_g A_g^{-1} B_g^{-1} = 1 \rangle$$

this flow has the following description in terms of generators:.

- $\Phi_t(\gamma) = \rho(\gamma)$ is constant if γ is either A_i for $1 \le i \le g$ or B_i for $2 \le i \le g$.
- $\Phi_t(B_1) = \rho(B_1)\zeta(t)$.
- Similar construction when γ separates..

Generalized twist deformations

- When α is a simple closed curve, then a flow Φ_t on $\mathrm{Hom}(\pi, G)$ exists, which covers the (local) flow of the Hamiltonian vector field $\mathrm{Ham}(f_\alpha)$.
- Example: α nonseparating curve A_1 in

$$\pi = \langle A_1, B_1, \dots, A_g, B_g \mid A_1 B_1 A_1^{-1} B_1^{-1} \dots, A_g B_g A_g^{-1} B_g^{-1} = 1 \rangle$$

  this flow has the following description in terms of generators:.
- $\Phi_t(\gamma) = \rho(\gamma)$ is constant if γ is either A_i for $1 \leq i \leq g$ or B_i for $2 \leq i \leq g$.
- $\Phi_t(B_1) = \rho(B_1)\zeta(t)$.
- Similar construction when γ separates..

Generalized twist deformations

- When α is a simple closed curve, then a flow Φ_t on $\mathrm{Hom}(\pi, G)$ exists, which covers the (local) flow of the Hamiltonian vector field $\mathrm{Ham}(f_\alpha)$.
- Example: α nonseparating curve A_1 in

$$\pi = \langle A_1, B_1, \dots, A_g, B_g \mid A_1 B_1 A_1^{-1} B_1^{-1} \dots, A_g B_g A_g^{-1} B_g^{-1} = 1 \rangle$$

this flow has the following description in terms of generators:.

- $\Phi_t(\gamma) = \rho(\gamma)$ is constant if γ is either A_i for $1 \leq i \leq g$ or B_i for $2 \leq i \leq g$.
- $\Phi_t(B_1) = \rho(B_1)\zeta(t)$.
- Similar construction when γ separates..

Generalized twist deformations

- When α is a simple closed curve, then a flow Φ_t on $\mathrm{Hom}(\pi, G)$ exists, which covers the (local) flow of the Hamiltonian vector field $\mathrm{Ham}(f_\alpha)$.
- Example: α nonseparating curve A_1 in

$$\pi = \langle A_1, B_1, \dots, A_g, B_g \mid A_1 B_1 A_1^{-1} B_1^{-1} \dots, A_g B_g A_g^{-1} B_g^{-1} = 1 \rangle$$

  this flow has the following description in terms of generators:.
- $\Phi_t(\gamma) = \rho(\gamma)$ is constant if γ is either A_i for $1 \leq i \leq g$ or B_i for $2 \leq i \leq g$.
- $\Phi_t(B_1) = \rho(B_1)\zeta(t)$.
- Similar construction when γ separates..

Generalized twist deformations

- When α is a simple closed curve, then a flow Φ_t on $\mathrm{Hom}(\pi, G)$ exists, which covers the (local) flow of the Hamiltonian vector field $\mathrm{Ham}(f_\alpha)$.
- Example: α nonseparating curve A_1 in

$$\pi = \langle A_1, B_1, \ldots, A_g, B_g \mid A_1 B_1 A_1^{-1} B_1^{-1} \ldots, A_g B_g A_g^{-1} B_g^{-1} = 1\rangle$$

  this flow has the following description in terms of generators:.
- $\Phi_t(\gamma) = \rho(\gamma)$ is constant if γ is either A_i for $1 \leq i \leq g$ or B_i for $2 \leq i \leq g$.
- $\Phi_t(B_1) = \rho(B_1)\zeta(t)$.
- Similar construction when γ separates..

Generalized twist deformations

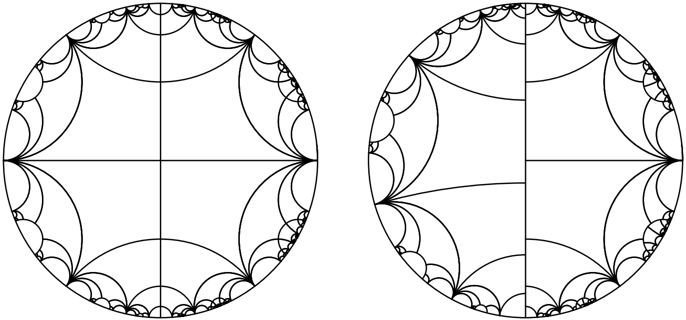
- When α is a simple closed curve, then a flow Φ_t on $\mathrm{Hom}(\pi, G)$ exists, which covers the (local) flow of the Hamiltonian vector field $\mathrm{Ham}(f_\alpha)$.
- Example: α nonseparating curve A_1 in

$$\pi = \langle A_1, B_1, \dots, A_g, B_g \mid A_1 B_1 A_1^{-1} B_1^{-1} \dots, A_g B_g A_g^{-1} B_g^{-1} = 1 \rangle$$

this flow has the following description in terms of generators:.

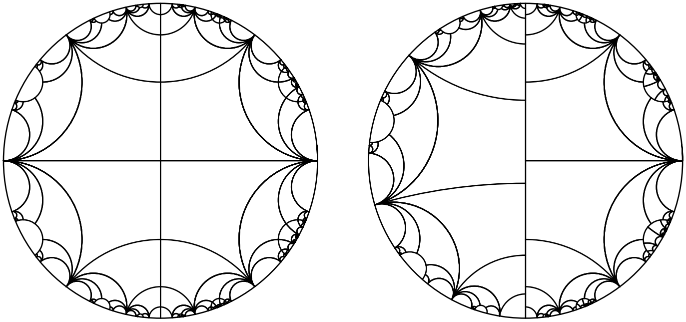
- $\Phi_t(\gamma) = \rho(\gamma)$ is constant if γ is either A_i for $1 \le i \le g$ or B_i for $2 \le i \le g$.
- $\Phi_t(B_1) = \rho(B_1)\zeta(t)$.
- Similar construction when γ separates..

Some earthquake deformations in the universal covering



A deformation of a hyperbolic structure supported on a closed geodesic. The developing map changes *discontinuously*, remaining equivariant under a varying family of holonomy representations.

Some earthquake deformations in the universal covering



A deformation of a hyperbolic structure supported on a closed geodesic. The developing map changes *discontinuously*, remaining equivariant under a varying family of holonomy representations.

Relation with the Mod(Σ)-action

- For $G = \mathrm{SU}(2)$, the equivalence relation generated by Dehn twists about *simple closed curves* is measurably equal to that given by twist flows (modulo Lebesgue nullsets).
- Because generically, the (discrete) Dehn twist is an irrational rotation of the circle (the trajectory of the twist flow).
- Since the Hamiltonian potentials (traces of simple loops) generate the character ring, their Hamiltonian flows generate a transitive action.
- Since the Dehn twists generate $\mathrm{Mod}(\Sigma)$, the action is ergodic on each connected component.
- Ergodic on each component when G is compact (G, Pickrell-Xia).

Relation with the Mod(Σ)-action

- For $G = \mathrm{SU}(2)$, the equivalence relation generated by Dehn twists about *simple closed curves* is measurably equal to that given by twist flows (modulo Lebesgue nullsets).
- Because generically, the (discrete) Dehn twist is an irrational rotation of the circle (the trajectory of the twist flow).
- Since the Hamiltonian potentials (traces of simple loops) generate the character ring, their Hamiltonian flows generate a transitive action.
- Since the Dehn twists generate Mod(Σ), the action is ergodic on each connected component.
- Ergodic on each component when G is compact (G, Pickrell-Xia).

Relation with the Mod(Σ)-action

- For $G = \mathrm{SU}(2)$, the equivalence relation generated by Dehn twists about *simple closed curves* is measurably equal to that given by twist flows (modulo Lebesgue nullsets).
- Because generically, the (discrete) Dehn twist is an irrational rotation of the circle (the trajectory of the twist flow).
- Since the Hamiltonian potentials (traces of simple loops) generate the character ring, their Hamiltonian flows generate a transitive action.
- Since the Dehn twists generate Mod(Σ), the action is ergodic on each connected component.
- Ergodic on each component when G is compact (G, Pickrell-Xia).

Relation with the Mod(Σ)-action

- For $G = \mathrm{SU}(2)$, the equivalence relation generated by Dehn twists about *simple closed curves* is measurably equal to that given by twist flows (modulo Lebesgue nullsets).
- Because generically, the (discrete) Dehn twist is an irrational rotation of the circle (the trajectory of the twist flow).
- Since the Hamiltonian potentials (traces of simple loops) generate the character ring, their Hamiltonian flows generate a transitive action.
- Since the Dehn twists generate Mod(Σ), the action is ergodic on each connected component.
- Ergodic on each component when G is compact (G, Pickrell-Xia).

Relation with the Mod(Σ)-action

- For $G = \mathrm{SU}(2)$, the equivalence relation generated by Dehn twists about *simple closed curves* is measurably equal to that given by twist flows (modulo Lebesgue nullsets).
- Because generically, the (discrete) Dehn twist is an irrational rotation of the circle (the trajectory of the twist flow).
- Since the Hamiltonian potentials (traces of simple loops) generate the character ring, their Hamiltonian flows generate a transitive action.
- Since the Dehn twists generate Mod(Σ), the action is ergodic on each connected component.
- Ergodic on each component when G is compact (G, Pickrell-Xia).

Relation with the Mod(Σ)-action

- For $G = \mathrm{SU}(2)$, the equivalence relation generated by Dehn twists about *simple closed curves* is measurably equal to that given by twist flows (modulo Lebesgue nullsets).
- Because generically, the (discrete) Dehn twist is an irrational rotation of the circle (the trajectory of the twist flow).
- Since the Hamiltonian potentials (traces of simple loops) generate the character ring, their Hamiltonian flows generate a transitive action.
- Since the Dehn twists generate $\mathrm{Mod}(\Sigma)$, the action is ergodic on each connected component.
- Ergodic on each component when G is compact (G, Pickrell-Xia).

Twist and bulging deformations for $\mathbb{RP}^2$-structures

- When $G = \mathrm{SL}(3,\mathbb{R})$, the generic centralizer is conjugate to the subgroup of diagonal matrices

$$\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

- which has two one-parameter subgroups:

$$\begin{bmatrix} e^t & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-t} \end{bmatrix}, \quad e^{-t/3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Twist and bulging deformations for $\mathbb{RP}^2$-structures

- When $G = \mathrm{SL}(3,\mathbb{R})$, the generic centralizer is conjugate to the subgroup of diagonal matrices

$$\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

- which has two one-parameter subgroups:

$$\begin{bmatrix} e^t & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-t} \end{bmatrix}, \quad e^{-t/3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Twist and bulging deformations for $\mathbb{RP}^2$-structures

- When $G = \mathrm{SL}(3, \mathbb{R})$, the generic centralizer is conjugate to the subgroup of diagonal matrices

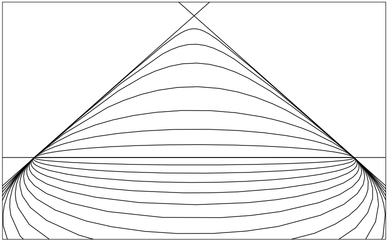
$$\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

- which has two one-parameter subgroups:

$$\begin{bmatrix} e^t & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-t} \end{bmatrix}, \quad e^{-t/3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

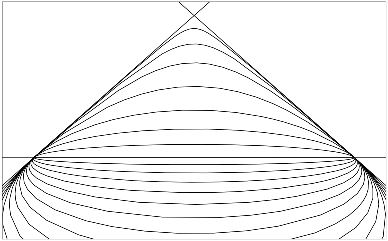
Bulging conics along a triangle in $\mathbb{RP}^2$

If Ω is the interior of an invariant conic, the the first one-parameter subgroup preserves Ω. The second one-parameter changes Ω by *bulging* it along a triangle tangent to $\partial\Omega$.



Bulging conics along a triangle in $\mathbb{RP}^2$

If Ω is the interior of an invariant conic, the the first one-parameter subgroup preserves Ω. The second one-parameter changes Ω by *bulging* it along a triangle tangent to $\partial\Omega$.



Bulging domains in $\mathbb{RP}^2$

- Start with a strictly convex C^1 domain Ω (like a disc). Each geodesic embeds in a triangle tangent to $\partial\Omega$.
- Choose disjoint lines $\Lambda \subset \Omega$, with instructions how to deform along Λ: (for each line $\lambda \subset \Lambda$, a one-parameter subgroup preserving λ.
- Fixing a basepoint in the complement of Λ, bulge/earthquake the curve inside the triangles tangent to $\partial\Omega$.
- Obtain a sequence of *piecewise conics* converging to the limit curve.

Bulging domains in $\mathbb{RP}^2$

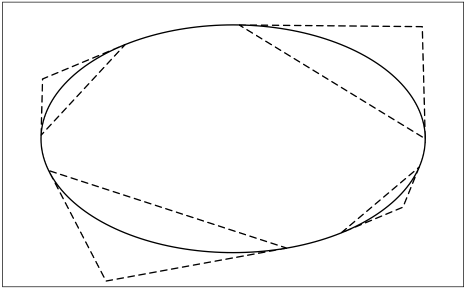
- Start with a strictly convex C^1 domain Ω (like a disc). Each geodesic embeds in a triangle tangent to $\partial\Omega$.
- Choose disjoint lines $\Lambda \subset \Omega$, with instructions how to deform along Λ: (for each line $\lambda \subset \Lambda$, a one-parameter subgroup preserving λ.
- Fixing a basepoint in the complement of Λ, bulge/earthquake the curve inside the triangles tangent to $\partial\Omega$.
- Obtain a sequence of *piecewise conics* converging to the limit curve.

Bulging domains in $\mathbb{RP}^2$

- Start with a strictly convex C^1 domain Ω (like a disc). Each geodesic embeds in a triangle tangent to $\partial\Omega$.
- Choose disjoint lines $\Lambda \subset \Omega$, with instructions how to deform along Λ: (for each line $\lambda \subset \Lambda$, a one-parameter subgroup preserving λ.
- Fixing a basepoint in the complement of Λ, bulge/earthquake the curve inside the triangles tangent to $\partial\Omega$.
- Obtain a sequence of *piecewise conics* converging to the limit curve.

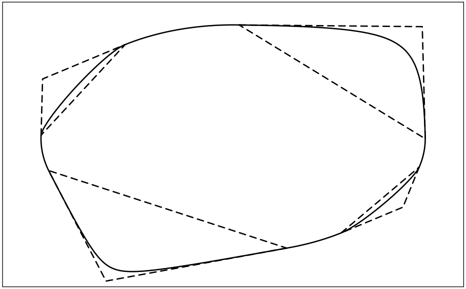
Bulging domains in $\mathbb{RP}^2$

- Start with a strictly convex C^1 domain Ω (like a disc). Each geodesic embeds in a triangle tangent to $\partial\Omega$.
- Choose disjoint lines $\Lambda \subset \Omega$, with instructions how to deform along Λ: (for each line $\lambda \subset \Lambda$, a one-parameter subgroup preserving λ.
- Fixing a basepoint in the complement of Λ, bulge/earthquake the curve inside the triangles tangent to $\partial\Omega$.
- Obtain a sequence of *piecewise conics* converging to the limit curve.

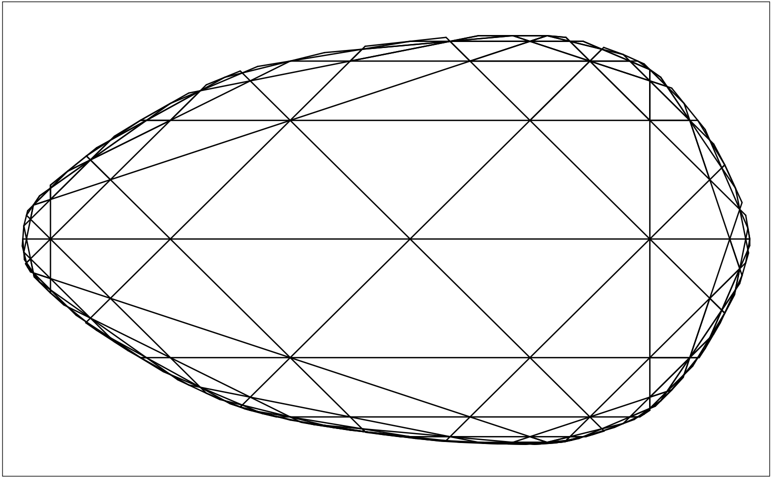


Bulging domains in $\mathbb{RP}^2$

- Start with a strictly convex C^1 domain Ω (like a disc). Each geodesic embeds in a triangle tangent to $\partial\Omega$.
- Choose disjoint lines $\Lambda \subset \Omega$, with instructions how to deform along Λ: (for each line $\lambda \subset \Lambda$, a one-parameter subgroup preserving λ.
- Fixing a basepoint in the complement of Λ, bulge/earthquake the curve inside the triangles tangent to $\partial\Omega$.
- Obtain a sequence of *piecewise conics* converging to the limit curve.



A domain in $\mathbb{RP}^2$ covering a closed surface



Iterated bulging of convex domains in $\mathbb{RP}^2$: Speculation

- If Ω covers a closed convex $\mathbb{RP}^2$-surface with $\chi < 0$, then $\partial\Omega$ is obtained from a conic by iterated bulgings and earthquakes.
- Is every properly convex domain $\Omega \subset \mathbb{RP}^2$ with strictly convex C^1 boundary obtained by iterated bulging-earthquaking?
- Thurston proved that any two marked hyperbolic structures on Σ can be related by (left)-earthquake along a unique *measured geodesic lamination.* Generalize this to convex $\mathbb{RP}^2$-structures.

Iterated bulging of convex domains in $\mathbb{RP}^2$: Speculation

- If Ω covers a closed convex $\mathbb{RP}^2$-surface with $\chi < 0$, then $\partial\Omega$ is obtained from a conic by iterated bulgings and earthquakes.
- Is every properly convex domain $\Omega \subset \mathbb{RP}^2$ with strictly convex C^1 boundary obtained by iterated bulging-earthquaking?
- Thurston proved that any two marked hyperbolic structures on Σ can be related by (left)-earthquake along a unique *measured geodesic lamination*. Generalize this to convex $\mathbb{RP}^2$-structures.

Iterated bulging of convex domains in $\mathbb{RP}^2$: Speculation

- If Ω covers a closed convex $\mathbb{RP}^2$-surface with $\chi < 0$, then $\partial\Omega$ is obtained from a conic by iterated bulgings and earthquakes.
- Is every properly convex domain $\Omega \subset \mathbb{RP}^2$ with strictly convex C^1 boundary obtained by iterated bulging-earthquaking?
- Thurston proved that any two marked hyperbolic structures on Σ can be related by (left)-earthquake along a unique *measured geodesic lamination.* Generalize this to convex $\mathbb{RP}^2$-structures.

Iterated bulging of convex domains in $\mathbb{RP}^2$: Speculation

- If Ω covers a closed convex $\mathbb{RP}^2$-surface with $\chi < 0$, then $\partial\Omega$ is obtained from a conic by iterated bulgings and earthquakes.
- Is every properly convex domain $\Omega \subset \mathbb{RP}^2$ with strictly convex C^1 boundary obtained by iterated bulging-earthquaking?
- Thurston proved that any two marked hyperbolic structures on Σ can be related by (left)-earthquake along a unique *measured geodesic lamination.* Generalize this to convex $\mathbb{RP}^2$-structures.