QUIZ 8 SOLUTIONS

Sections at 12 pm

Problem 1

Compute all fourth roots of unity.

Solution. For each natural number n there are exactly n n-th roots of unity, which can be expressed as:

$$z_k = e^{i\frac{2\pi}{n}k},\ k = 0, 1, \dots, n-1.$$

In our case n=4, hence,

$$z_k = e^{i\frac{2\pi}{4}k} = e^{i\frac{\pi k}{2}},\ k = 0, 1, 2, 3.$$

Thus, the 4-th roots of 1 are:

$$z_0 = e^{i0} = 1,\ z_1 = e^{i\frac{2\pi}{4}} = i,\ z_2 = e^{i\frac{2\pi}{4}2} = e^{i\pi} = -1,\ z_3 = e^{i\frac{2\pi}{4}3} = e^{i\frac{3\pi}{2}} = -i.$$

Answer: 1, i, -1, -i.

Problem 2

Find the complex polar represntation of the complex number $z = -1 + i\sqrt{3}$.

Solution. We need to find such R and Θ that $z = -1 + i\sqrt{3} = Re^{i\Theta}$.

$$R = |z| = \sqrt{1^2 + (\sqrt{3})^2} = \sqrt{4} = 2.$$

Therefore,

$$z = 2e^{i\Theta} = 2(-\frac{1}{2} + i\frac{\sqrt{3}}{2}),$$

where Θ satisfies

$$\cos\Theta = -\frac{1}{2}, \sin\Theta = \frac{\sqrt{3}}{2} \Rightarrow \Theta = \frac{2\pi}{3}.$$

Answer: $z = 2e^{i\frac{2\pi}{3}}$.

Problem 3

Write in polar coordinates the equation

$$y^2 = 4.$$

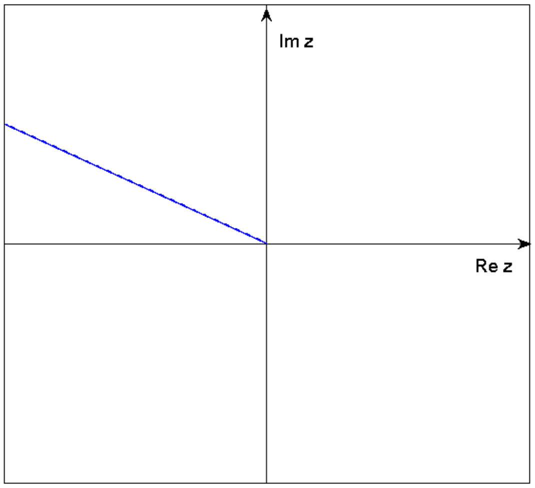
Solution. In polar coordinates $y = r\sin\Theta$. Hence,

$$r^2\sin^2\Theta = 4, \quad \sin\Theta = \pm 2.$$

Problem 4

Sketch the graph of the polar equation

$$\Theta = -\frac{7\pi}{6}$$



SOLUTION. Since there is no restriction on r, it can take all possible values, i.e. r can be any non-negative number. Hence, the graph should contain all complex numbers $z = re^{-i\frac{7\pi}{6}}$, those are shown in blue on the plot.

Sections at 1 pm

PROBLEM 1

Compute all fourth roots of unity.

SOLUTION. For each natural number n there are exactly n n-th roots of unity, which can be expressed as:

$$z_k = e^{i\frac{2\pi}{n}k},\ k = 0, 1, \ldots, n-1.$$

In our case n=4, hence,

$$z_k = e^{i\frac{2\pi}{4}k} = e^{i\frac{\pi k}{2}},\ k = 0, 1, 2, 3.$$

Thus, the 4-th roots of 1 are:

$$z_0 = e^{i0} = 1,\ z_1 = e^{i\frac{2\pi}{4}} = i,\ z_2 = e^{i\frac{2\pi}{4}2} = e^{i\pi} = -1,\ z_3 = e^{i\frac{2\pi}{4}3} = e^{i\frac{3\pi}{2}} = -i.$$

ANSWER: 1, i, -1, $-i$.

PROBLEM 2

Find the complex polar represntation of the complex number $z = 2 + 2i$.

SOLUTION. We need to find such R and Θ that $z = 2 + i2 = Re^{i\Theta}$.

$$R = |z| = \sqrt{2^2 + 2^2} = \sqrt{8} = 2\sqrt{2}.$$

Therefore,

$$z = 2\sqrt{2}e^{i\Theta} = 2\sqrt{2}(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}),$$

where Θ satisfies

$$\cos\Theta = \frac{1}{\sqrt{2}}, \sin\Theta = \frac{1}{\sqrt{2}} \Rightarrow \Theta = \frac{\pi}{4}.$$

ANSWER: $z = 2\sqrt{2}e^{i\pi/4}$.

PROBLEM 3
Write in polar coordinates the equation

$$2x + 3y = 4.$$

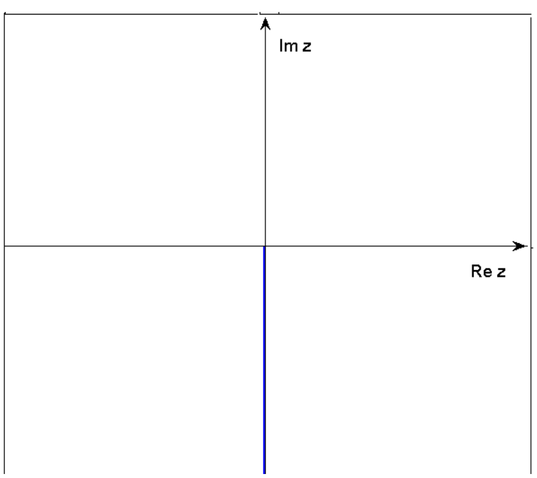
SOLUTION. In polar coordinates $x = r\cos\Theta$, $y = r\sin\Theta$. Hence,

$$2r\cos(\Theta) + 3r\sin(\Theta) = 4.$$

PROBLEM 4
Sketch the graph of the polar equation

$$\Theta = \frac{3\pi}{2}$$

SOLUTION. Since there is no restriction on r, it can take all possible values, i.e. r can be any non-negative number. Hence, the graph should contain all complex numbers $z = re^{i3\pi/2}$. This set is shown in blue on the plot below:



Sections at 2 pm

PROBLEM 1
Compute all fifth roots of unity.
SOLUTION. For each natural number n there are exactly n n-th roots of unity, which can be expressed as:

$$z_k = e^{i\frac{2\pi}{n}k},\ k = 0, 1, \ldots, n-1.$$

In our case n=5, hence,

$$z_k = e^{i\frac{2\pi}{5}k},\ k = 0, 1, 2, 3, 4.$$

Thus, the 5-th roots of 1 are:

$$z_0 = e^{i0} = 1,\ z_1 = e^{i\frac{2\pi}{5}},\ z_2 = e^{i\frac{4\pi}{5}},\ z_3 = e^{i\frac{6\pi}{5}},\ z_4 = e^{i\frac{8\pi}{5}}.$$

Answer: $1,\ i,\ -1,\ -i$.

Problem 2

Find the complex polar representation of the complex number $z = 1 + \sqrt{3}i$.

Solution. We need to find such R and Θ that $z = 1 + \sqrt{3}i = Re^{i\Theta}$.

$$R = |z| = \sqrt{1^2 + (\sqrt{3})^2} = \sqrt{4} = 2.$$

Therefore,

$$z = 2e^{i\Theta} = 2(\frac{1}{2} + i\frac{\sqrt{3}}{2}),$$

where Θ satisfies

$$\cos\Theta = \frac{1}{2}, \sin\Theta = \frac{\sqrt{3}}{2} \Rightarrow \Theta = \frac{\pi}{3}.$$

Answer: $z = 2e^{i\pi/3}$.

Problem 3

Write in polar coordinates the equation

$$x^2 + y^2 = 1.$$

Solution. In polar coordinates $x = r\cos\Theta$, $y = r\sin\Theta$. Hence,

$$r^2\cos^2(\Theta) + r^2\sin^2(\Theta) = r^2 = 1$$

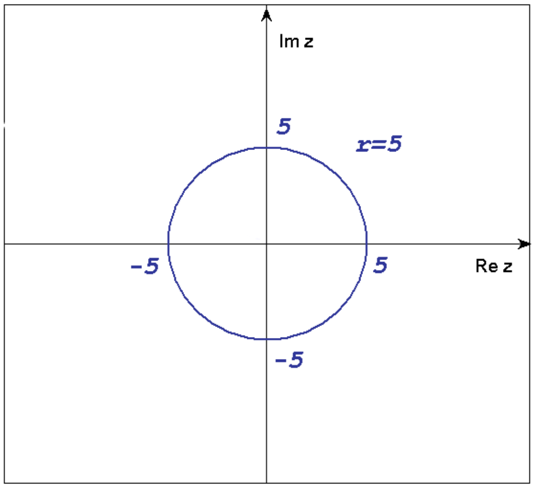
Answer: $r^2 = 1$.

Problem 4

Sketch the graph of the polar equation

$$r = 5$$

Solution. Since there is no restriction on Θ, it can take all possible values. Hence, the graph should contain all complex numbers $z = 5e^{i\Theta}$, i.e. all numbers with $|z| = 5$. This set is shown in blue on the plot.



Sections at 3 pm

PROBLEM 1

Compute all fifth roots of unity.

SOLUTION. For each natural number n there are exactly n n-th roots of unity, which can be expressed as:

$$z_k = e^{i\frac{2\pi}{n}k},\ k = 0, 1, \dots, n-1.$$

In our case n=5, hence,

$$z_k = e^{i\frac{2\pi}{5}k},\ k = 0, 1, 2, 3, 4.$$

Thus, the 5-th roots of 1 are:

$$z_0 = e^{i0} = 1,\ z_1 = e^{i\frac{2\pi}{5}} = i,\ z_2 = e^{i\frac{4\pi}{5}},\ z_3 = e^{i\frac{6\pi}{5}},\ z_4 = e^{i\frac{8\pi}{5}}.$$

ANSWER: $1,\ i,\ -1,\ -i$.

PROBLEM 2

Find the complex polar representation of the complex number $z = 3 + 3i$.

SOLUTION. We need to find such R and Θ that $z = 3 + 3i = Re^{i\Theta}$.

$$R = |z| = \sqrt{3^2 + 3^2} = \sqrt{18} = 3\sqrt{2}.$$

Therefore,

$$z = 2e^{i\Theta} = 2(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}),$$

where Θ satisfies

$$\cos\Theta = \frac{1}{\sqrt{2}}, \sin\Theta = \frac{1}{\sqrt{2}} \Rightarrow \Theta = \frac{\pi}{4}.$$

ANSWER: $z = 3\sqrt{2}e^{i\pi/4}$.

PROBLEM 3

Write in polar coordinates the equation

$$x^2 + y^2 = 4y.$$

SOLUTION. In polar coordinates $x = r\cos\Theta$, $y = r\sin\Theta$. Hence,

$$r^2\cos^2(\Theta) + r^2\sin^2(\Theta) = r^2 = 4r\sin(\Theta)$$

ANSWER: $r^2 = 4r\sin(\Theta)$.

PROBLEM 4

Sketch the graph of the polar equation

$$r = 5$$

SOLUTION. Since there is no restriction on Θ, it can take all possible values. Hence, the graph should contain all complex numbers $z = 7e^{i\Theta}$, i.e. all numbers with $|z| = 7$. This set is shown in blue on the plot.