MATH 141, FALL 2015, Example of 1st Order Linear Differential Equation

Solve the following 1st order linear differential equation:

$$\frac{dy}{dx} + (2x+3)y = 2x+3. \tag{1}$$

We start by computing:

$$S(x) = \int (2x+3)\,dx = x^2+3x+C. \tag{2}$$

Then, the formula from our textbook tells us that a solution of the differential equation (1) will have the form:

$$y = e^{-S(x)} \int e^{S(x)} Q(x)\,dx.$$

Remark 0.1. *Note that in the above equation we cannot cancel out $e^{-S(x)}$ with $e^{S(x)}$. One of ways to avoid this mistake is by switching to a different variable for the integral. However remember that at the end you* **must** *return to the x variable. Alternatively, to avoid the temptation of oversimplifying the integration by canceling the exponentials, we may solve the integral separately, as is shown below.*

Let us solve first the integral component, by substituting S from (2):

$$\int e^{S(x)} Q(x)\,dx = \int e^{x^2+3x+C}(2x+3)\,dx = e^C \int e^{x^2+3x}(2x+3)\,dx.$$

We use the substitution $t = x^2+3x$, $dt = 2x+3$, to obtain

$$\int e^{x^2+3x}(2x+3)\,dx = \int e^t\,dt = e^t + C' = e^{x^2+3x} + C'.$$

Hence,

$$y = e^{-x^2-3x-C}\left(e^C(e^{x^2+3x} + C')\right).$$

Please note that now e^{-C} and e^C multiply out to 1, so that:

$$y = e^{-x^2-3x}\left(e^{x^2+3x} + C'\right).$$

Furthermore, we can simplify this to:

$$y = 1 + C'e^{-x^2-3x}.$$

Remark 0.2. *One can always just use C to denote a generic constant at this point:*

$$y = 1 + Ce^{-x^2-3x}.$$

Finally, we can verify that this is indeed a solution of (1), by differentiating y:

$$\frac{dy}{dx} = Ce^{-x^2-3x}(-2x-3),$$

and plugging it into equation (1):

$$Ce^{-x^2-3x}(-2x-3) + (2x+3)\left(1 + Ce^{-x^2-3x}\right) = 2x+3.$$