MATH 416, extra problems

1. Suppose a divides b and b divides a. Must $a = b$?

2. Suppose a divides b and b divides c. Must a divide c?

3. Suppose a^2 divides b^2. Must a divide b?

4. Prove that if $p^2 \in \mathbb{Z}$ is a prime number, then $\sqrt{p}$ is not a rational number.

5. Let $N > 0$ be given and suppose $x, y \in \mathbb{R}^N$ satisfy $\|x\|_2 = \|y\|_2 = 1$.

a) Find x such that $\|x\|_1$ is as large as possible.

b) Find y such that $\|y\|_\infty$ is as small as possible.

6. Prove that $\|x - y\| \geq |\|x\| - \|y\||$ for any vectors x, y in a normed vector space X.

7. Find the 1-periodization of the function $f(x) = e^{-|x|}$.

8. For real $\epsilon > 0$ and α, define the dilation operator D_ϵ and the translation operator T_α, which act on functions $f = f(t)$ of one real variable as follows:

$$T_\alpha(u)(t) = u(t - \alpha) \quad D_\epsilon(u)(t) = \epsilon^{-1/2} u(t/\epsilon).$$

a) Show that these are linear transformations with inverses $T_\alpha^{-1} = T_{-\alpha}$ and $D_\epsilon^{-1} = D_{1/\epsilon}$

b) Compute the composition $T_\alpha(D_\epsilon(F))$ for a function $F = F(x)$.

9. Show that the set of functions $\{1, \sqrt{2}\sin(2\pi nt), \sqrt{2}\cos(2\pi nt) : n = 1, 2, 2 \ldots\}$ is orthonormal with respect to the Hermitean inner product.

10. Show that the set of functions $\{\sqrt{2}\sin(2\pi nt) : n = 1, 2, 3, \ldots\}$ is orthonormal with respect to the real inner product.

11. Show that the set of functions $\{1, \sqrt{2}\cos(2\pi nt) : n = 1, 2, 3, \ldots\}$ is orthonormal with respect to the real inner product.

12. Compute the sine-cosine Fourier series of the 1-periodic function $f(x) = \cos^2(2\pi x)$.

13. Compute the complex exponential Fourier series of the 1-periodic function $\sin(2\pi kt - d)$, where d is a constant real number, and k is an integer.

14. Write out explicitly the matrices for the 2×2 and 4×4 Discrete Fourier and Hartley transforms (F2, F4, H2 and H4).

15. Write out explicitly the matrices for the 1×1 Discrete Fourier, Hartley, and DCT-IV transforms (F1, H1 and CIV1).

16. Compute the complex exponential Fourier coefficients of the function $e^{i\alpha x} f(x)$, $x \in [0, 1]$, in terms of the complex exponential Fourier coefficients of the function $f(x)$.