MATH 141, FALL 2015, **A note on Limit Comparison Test**

Limit Comparison Test (Theorem 9.14) can be stated a little differently than it is in the textbook.

Theorem 0.1 (Limit Comparison Test)**.** Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty}$ be positivie series $(a_n, b_n > 0)$. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L,$$

where $0 < L < \infty$, then either both series converge, or both series diverge.

The above result is equivalent to Theorem 9.14, as one easily notices that $0 < L < \infty$ if and only if $0 < \frac{1}{L} < \infty$. As such, it is not allowing us to analyze any new examples. The major reson for stating this above formulation is to show that the roles of numerator and denominator are interchangeable in LCT.

The next result, however, is a new observation, for which the roles of a_n and b_n are, clearly, no longer interchangeable.

Theorem 0.2 (Limit Comparison Test 2)**.** Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be positivie series$(a_n, b_n > 0)$. Assume

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L.$$

a) If $L = 0$ and $\sum_{n=1}^{\infty} b_n$ converges then $\sum_{n=1}^{\infty} a_n$ also converges.
b) If $L = \infty$ and $\sum_{n=1}^{\infty} b_n$ diverges then $\sum_{n=1}^{\infty} a_n$ also diverges.

Example 0.3. Let us consider the following series:

$$\sum_{n=1}^{\infty} \frac{1}{n^2 \ln(n)}.$$

One is, naturally, tempted to compare this to the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2},$$

which, as we know, converges. However, it is easy to see that

$$\frac{1/n^2}{1/(n^2 \ln(n))} = \frac{n^2 \ln(n)}{n^2} \to \infty,$$

and

$$\frac{1/(n^2 \ln(n))}{1/n^2} = \frac{n^2}{n^2 \ln(n)} \to 0,$$

as $n \to \infty$. Thus, the Limit Comparison Theorem from the textbook does not work anymore. But part (a) of the Limit Comparison Test 2 will provide us with an answer. Indeed, observe that from the last limit above, we can conclude that the series $\sum 1/(n^2 \ln(n))$ converges, because $\sum 1/n^2$ converges.

Example 0.4. Similarly, we can consider the following series:

$$\sum_{n=1}^{\infty} \frac{\ln(n)}{n}.$$

Since,

$$\frac{\ln(n)/n}{1/n} = \ln(n) \to \infty,$$

and because $\sum 1/n$ diverges, from part (b) of the Limit Comparison Test 2, we can conclude that $\sum \ln(n)/n$ also diverges.