MATH 141, FALL 2014, **A complete non-trivial example of partial fractions**

Consider the following integral

$$I = \int \frac{x^5 + 4x^4 + 7x^3 + 12x^2 + 13x + 7}{x^4 + 4x^3 + 5x^2 + 4x + 4}\, dx$$

As the degree of the numerator is larger than the degree of the denominator, we must divide the first by the latter:

$$x^5 + 4x^4 + 7x^3 + 12x^2 + 13x + 7 \ : \ x^4 + 4x^3 + 5x^2 + 4x + 4,$$

and we note that the quotient is x and the reminder is $2x^3 + 8x^2 + 9x + 7$. Hence,

$$I = \int x\, dx + \int \frac{2x^3 + 8x^2 + 9x + 7}{x^4 + 4x^3 + 5x^2 + 4x + 4}\, dx = \frac{1}{2}x^2 + \int \frac{2x^3 + 8x^2 + 9x + 7}{x^4 + 4x^3 + 5x^2 + 4x + 4}\, dx.$$

Next, we need to identify the factorization of the denominator. As it is a polynomial with integer coefficients, and the leading coefficient being 1, we should look for one of its roots to be a divisor of the constant term. Thanks to the factor theorem, we thus conclude that $x+2$ must be a factor of the denominator. Dividing $x^4+4x^3+5x^2+4x+4$ by $x+2$ results in x^3+2x^2+x+2. We again look for its roots among the divisors of the constant term, and note that $x+2$ must be a factor of x^3+2x^2+x+2. Dividing again by $x+2$ results in x^2+1. This process can be summarized as:

$$x^4 + 4x^3 + 5x^2 + 4x + 4 = (x + 2)^2(x^2 + 1).$$

Thus, we have factored the denominator completely, as Vieta's formulas tell us that for the polynomial x^2+1 its discriminant is -4, and thus this polynomial has no real roots.

The next step is to write the function

$$\frac{2x^3 + 8x^2 + 9x + 7}{(x + 2)^2(x^2 + 1)},$$

as a sum of its partial fractions:

$$\frac{A}{x + 2} + \frac{B}{(x + 2)^2} + \frac{Cx + D}{x^2 + 1}.$$

Thus, we need to solve the following system of equations:

$$\begin{aligned} A + C &= 2, \\ 2A + B + 4C + D &= 8, \\ A + 4C + 4D &= 9, \\ 2A + B + 4D &= 7. \end{aligned}$$

This system of equations is solved in the usual way and provides us with the following solution:

$$A = 1, \quad B = 1, \quad C = 1, \quad D = 1.$$

Therefore, we note that

$$I = \frac{1}{2}x^2 + \int \frac{2x^3 + 8x^2 + 9x + 7}{x^4 + 4x^3 + 5x^2 + 4x + 4}\, dx = \frac{1}{2}x^2 + \int \left(\frac{1}{x+2} + \frac{1}{(x+2)^2} + \frac{x+1}{x^2+1} \right)\, dx.$$

As we can integrate all the functions directly by taking antderivatives, we have

$$I = \frac{1}{2}x^2 + \ln(|x+2|) - \frac{1}{x+2} + \arctan(x) + C.$$