MATH 630, Spring 2007, SAMPLE FINAL

1) Let $f : [0,1] \to \mathbb{R}$ be an increasing right continuous function with the property that

$$\forall\, g \in C([0,1]), \quad \int_0^1 g\, df = 0.$$

Prove that f is a constant function.

2) Let $f : [a,b] \to \mathbb{R}$ be a differentiable function and suppose $f' \in BV([a,b])$. Prove that $f' \in C([a,b])$.

3) Let $(X, \mathcal{A}, \mu)$ be a measure space. Let $1 \le p \le \infty$, $1/p + 1/q = 1$. Assume that $\{f_n : n = 1, \dots\} \subseteq L^p_\mu(X)$, $\{g_n : n = 1, \dots\} \subseteq L^q_\mu(X)$ are such that $f_n \to f$ in $L^p_\mu(X)$ and $g_n \to g$ in $L^q_\mu(X)$. Prove that $f_n g_n \to fg$ in $L^1_\mu(X)$.

4) Let $(X, \mathcal{A}, \mu)$ be a finite measure space. Assume that $\{f_n : n = 1, \dots\} \subseteq L^{2007}_\mu(X)$ is bounded in the norm $L^{2007}_\mu(X)$ and that $f_n \to f$ μ-*a.e.* Show that $f_n \to f$ in $L^1_\mu(X)$.

5) Let $(X, \mathcal{A}, \mu)$ be a finite measure space. Show that $L^p_\mu(X) \subseteq L^r_\mu(X)$, for any $1 \le r \le p \le \infty$. Show that the assumption $\mu(X) < \infty$ is necessary.