Calculus 141, sections 9.8-9.9 Radius of Convergence Examples

notes by Tim Pilachowski, Fall 2008

Be sure to check out Theorem 9.24 in the text for information about radius of convergence and interval of convergence. See table 9.1 for examples.

Theory: We know about convergence for a geometric series. For $c \neq 0$ and $m \geq 0$, the geometric series $\sum_{n=m}^{\infty} c\,r^n$ converges if and only if $|r| < 1$, and in this case $\sum_{n=m}^{\infty} c\,r^n = \frac{c\,r^m}{1-r}$. For a power series, $\sum_{n=m}^{\infty} c_n x^n$, for which the coefficients form a sequence, one method will be to 1) rewrite it as a geometric series, 2) identify *c* and *r*, and 3) set up an inequality $|r| < 1$ to solve for values of *x*.

Example A: Find the interval of convergence of $\sum_{n=1}^{\infty} \frac{2^{n+2}}{3^n} x^{3n}$.

1) Rewrite this power series as a geometric series: $\sum_{n=1}^{\infty} \frac{2^{n+2}}{3^n} x^{3n} = \sum_{n=1}^{\infty} 2^2 \left(\frac{2}{3} x^3\right)^n$.

2) Identify $c = 4$ and $r = \frac{2}{3} x^3$.

3) Solve: $\left|\frac{2}{3} x^3\right| = \frac{2}{3}\left|x^3\right| < 1 \Rightarrow \left|x^3\right| < \frac{3}{2} \Rightarrow |x| < \sqrt[3]{\frac{3}{2}}$, so the radius of convergence $R = \sqrt[3]{\frac{3}{2}}$.

At the boundaries: $\sum_{n=1}^{\infty} 2^2 \left(\frac{2}{3} * \left[-\sqrt[3]{\frac{3}{2}}\right]^3\right)^n = \sum_{n=1}^{\infty} 2^2 (-1)^n$ and $\sum_{n=1}^{\infty} 2^2 \left(\frac{2}{3} * \left[\sqrt[3]{\frac{3}{2}}\right]^3\right)^n = \sum_{n=1}^{\infty} 2^2 (1)^n$, neither of which converge. Thus the interval of convergence is $\left(-\sqrt[3]{\frac{3}{2}}, \sqrt[3]{\frac{3}{2}}\right)$.

An advantage to a geometric series is that, within the radius of convergence, we can find the sum.

For this series, $\sum_{n=1}^{\infty} \frac{2^{n+2}}{3^n} x^{3n} = \sum_{n=1}^{\infty} 2^2 \left(\frac{2}{3} x^3\right)^n = \frac{c\,r^m}{1-r} = \frac{4\left(\frac{2}{3} x^3\right)^1}{1 - \frac{2}{3} x^3} = \frac{8x^3}{3 - 2x^3}$.

Theory: We can also apply the other tests for convergence to create an equation to solve for *x*.

Example B: Find the interval of convergence of $\sum_{n=0}^{\infty} \frac{2^n x^n}{n+1}$. This time the Ratio Test is suitable.

$$\lim_{n\to\infty} \left| \frac{\frac{2^{n+1} x^{n+1}}{n+2}}{\frac{2^n x^n}{n+1}} \right| = \lim_{n\to\infty} \left| \frac{2(n+1)x}{n+2} \right| = 2|x| < 1 \Rightarrow |x| < \frac{1}{2}, \text{ i.e. radius of convergence } R = \frac{1}{2}$$

At the boundaries: $\sum_{n=0}^{\infty} \frac{2^n}{n+1} \left(-\frac{1}{2}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$ which converges,

and $\sum_{n=0}^{\infty} \frac{2^n}{n+1} \left(\frac{1}{2}\right)^n = \sum_{n=0}^{\infty} \frac{(1)^n}{n+1} = \sum_{n=0}^{\infty} \frac{1}{n+1}$ which diverges. Thus the interval of convergence is $\left[-\frac{1}{2}, \frac{1}{2}\right)$.

Example C: Find the interval of convergence of $\sum_{n=0}^{\infty} \frac{2^n x^n}{n!}$.

Do this one for practice. Use the Ratio Test to show that radius of convergence = ∞ and the interval of convergence is $(-\infty, \infty)$.

Example D: Find the radius of convergence of $\sum_{n=0}^{\infty} \left(\frac{n+1}{n}\right)^{n^2} x^n$. The Root Test will work well here.

$$\lim_{n\to\infty} \sqrt[n]{\left|\left(\frac{n+1}{n}\right)^{n^2} x^n\right|} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n |x| = e|x| < 1 \quad \Rightarrow \quad \text{radius of convergence} = \frac{1}{e}.$$

Theory: The Lagrange Remainder Formula gives us another approach.

Example E: Show that the Taylor series generated by $f(x) = e^x$ about $x = 0$ converges to e^x for all x. In other words, show that the interval of convergence is $(-\infty, \infty)$.

For all orders of derivatives, $f^{(n)}(x) = e^x \Rightarrow f^{(n)}(0) = e^0 = 1$, so the Taylor series for $f(x)$ is

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

The Lagrange Remainder Formula gives us $r_n(x) = \frac{f^{(n+1)}(t_x)}{(n+1)!} x^{n+1} = \frac{e^{t_x}}{(n+1)!} x^{n+1}$.

We have three cases to consider:

I. When x is negative, so is t_x, and $0 < e^{t_x} < 1$.

II. When $x = 0$, $e^0 = 1$ and $r_n(x) = 0$.

III. When x is positive, so is t_x, and $e^{t_x} < e^x$.

Thus, for all three cases $0 \le |r_n(x)| = \left|\frac{e^{t_x}}{(n+1)!} x^{n+1}\right| < \frac{e^x}{(n+1)!}|x|^{n+1}$.

For any given value of x, the sequence $\left\{\frac{e^x}{(n+1)!}|x|^{n+1}\right\}$ is positive and decreasing.

So $\lim_{n\to\infty} \frac{e^x}{(n+1)!}|x|^{n+1} = 0$, thus $\lim_{n\to\infty} r_n(x) = 0$, and the Taylor series for e^x converges to e^x for all values of x.