Midterm 1 Problem 2.

Below are two possible ways of solving this problem. The first one is a "brute-force"-computational approach, the other one is based on noticing the relationship between $x(t)$ and $y(t)$ and changing the variable to simplfy the problem.

SOLUTION I

1 step (8 points)

Recall the formula for the area of a surface obtained by rotating a parametrically given curve $x = f(t), y = g(t), t \in [a, b]$ around a coordinate axis:

$$A = 2\pi \int_a^b y(t)\sqrt{x'(t)^2 + y'(t)^2}dt$$

2 step (4 points)

Find the derivatives:

$$x'(t) = y'(t) = ln(2)2^t$$

3 step (8 points)

Substitute and integrate:

$$A = 2\pi \int_0^1 2^t \sqrt{2(ln(2))^2 2^{2t}}dt = 2\pi\sqrt{2}ln(2) \int_0^1 2^{2t}dt$$

let $u = 2^t$ then $du = 2^t ln(2)dt$, hence,

$$A = 2\pi\sqrt{2} \int_1^2 udu = 2\pi\sqrt{2}\frac{u^2}{2}|_1^2 = 3\pi\sqrt{2}$$

SOLUTION II

1 step (8 points)

Notice that $f(t) = g(t)$, hence, $x = y$ for the entire range of values of 2^t when t changes from 0 to 1. Hence, the curve can be defined by

$$y = x,\ x = 2^t,\ t \in [0, 1], \quad \text{i.e. } y = x, x \in [1, 2]$$

2 step (6 points)

Recall the formula for the area of a surface obtained by rotating the graph of a function $y = F(x)$, $x \in [\alpha, \beta]$

$$A = 2\pi \int_\alpha^\beta F(x)\sqrt{1 + (F'(x))^2}dx$$

3 step (6 points)

Substitute $y = x$, $x \in [1, 2]$ in the above expression and evaluate the integral:

$$A = 2\pi \int_0^1 x\sqrt{1 + 1^2}dx = 2\pi\sqrt{2}\frac{x^2}{2}|_1^2 = 3\pi\sqrt{2}$$