Problem 2

Evaluate the following indefinite integrals

(a) $\int \log_6(x)\ dx$

(b) $\int x^3 \sin(x^2)\ dx$

Solution:

(a) Recall the change of base formula for logarithms:

$$\log_a(x) = \frac{\ln x}{\ln a}.$$

Then this integral becomes

$$\int \log_6(x)\ dx = \int \frac{\ln x}{\ln 6}\ dx = \frac{1}{\ln 6} \int \ln x\ dx. \qquad (3\ \text{pts})$$

Apply integration by parts using the following substitutions:

$$\begin{aligned} u &= \ln x, & dv &= dx; \\ du &= \tfrac{1}{x}\ dx, & v &= x. \end{aligned} \qquad (3\ \text{pts})$$

So by integration by parts:

$$\int \log_6(x)\ dx = \frac{1}{\ln 6}\left(x \ln x - \int dx\right) = \frac{1}{\ln 6}\left(x \ln x - x\right) + C. \qquad (4\ \text{pts})$$

Note: Omission of the arbitrary constant C results in a deduction of one point.

(b) There are two common ways to solve this problem: either do a change of variables (u-substitution) first and then do integration by parts, or do integration by parts first and then do a change of variables. We will outline both solutions.

(i) Begin with the following substitution:

$$\begin{aligned} u &= x^2, \\ du &= 2x\ dx. \end{aligned} \qquad (5\ \text{pts})$$

Then the integral becomes

$$\int x^3 \sin(x^2)\ dx = \frac{1}{2} \int u \sin(u)\ du.$$

Now we may use integration by parts. Let

$$\begin{aligned} v &= u, \quad & dw &= \sin u \ du; \\ dv &= du, \quad & w &= -\cos u. \end{aligned} \qquad (6 \ \text{pts})$$

So by the integration by parts formula:

$$\begin{aligned} \int x^3 \sin(x^2) \ dx &= \frac{1}{2}\left(-u\cos u + \int \cos u \ du\right) \\ &= \frac{1}{2}\left(-u\cos u + \sin u\right) + C \\ &= \frac{1}{2}\left(-x^2\cos(x^2) + \sin(x^2)\right) + C. \end{aligned} \qquad (4 \ \text{pts})$$

Note again that forgetting the constant C will result in a deduction of one point.

(ii) Alternatively begin with the following substitutions:

$$\begin{aligned} u &= x^2, \quad & dv &= x\sin(x^2) \ dx; \\ du &= 2x \ dx, \quad & v &= \textstyle\int x\sin(x^2) \ dx. \end{aligned} \qquad (6 \ \text{pts})$$

Notice that v here is not obvious. To solve explicitly let

$$w = x^2,$$

$$dw = 2x \ dx.$$

Then

$$v = \int x\sin(x^2) \ dx = \frac{1}{2}\int \sin w \ dw = \frac{-1}{2}\cos(x^2) + C. \qquad (5 \ \text{pts})$$

Hence

$$\begin{aligned} \int x^3 \sin(x^2) \ dx &= \frac{-x^2}{2}\cos(x^2) + \int x\cos(x^2) \ dx \\ &= \frac{1}{2}\left(-x^2\cos(x^2) + \sin(x^2)\right) + C. \end{aligned} \qquad (4 \ \text{pts})$$

As before, C is worth one point.