Problem 2

Evaluate the following integrals:

(a) $\int \frac{e^x}{\sqrt{1-e^{2x}}}\ dx$;

(b) $\int \sin^2(x)\cot^2(x)\tan(x)\sec^4(x)\ dx$.

Solution:

(a) Let

$$\begin{aligned} u &= e^x \\ du &= e^x\ dx. \end{aligned} \qquad (5\ \ \text{pts})$$

Then

$$\begin{aligned} \int \frac{e^x}{\sqrt{1-e^{2x}}}\ dx &= \int \frac{1}{\sqrt{1-u^2}}\ du \\ &= \sin^{-1}(u) + C & (3\ \ \text{pts}) \\ &= \sin^{-1}(e^x) + C. & (2\ \ \text{pts}) \end{aligned}$$

One point will be deducted for omission of the arbitrary constant C.

Note that this problem can also be solved by the method of trigonometric substitution, but the result should be the same.

(b) Begin by turning all of the trig functions in the integrand into powers of sine and cosine:

$$\int \sin^2(x)\cot^2(x)\tan(x)\sec^4(x)\ dx = \int \frac{\sin(x)}{\cos^3(x)}\ dx. \qquad (5\ \ \text{pts})$$

There are three ways to proceed from here, all of which are scored identically: 5 points for the proper substitution, and 5 points for integrating and arriving at the answer correctly. As usual one point is deducted for omission of the arbitrary constant C.

(1) Let

$$\begin{aligned} u &= \cos(x) \\ du &= -\sin(x)\ dx. \end{aligned} \qquad (5\ \ \text{pts})$$

Then

$$\begin{aligned}
\int \frac{\sin(x)}{\cos^3(x)}\ dx &= -\int \frac{1}{u^3}\ du \\
&= \frac{1}{2u^2} + C \\
&= \frac{1}{2\cos^2(x)} + C \\
&= \frac{\sec^2(x)}{2} + C.
\end{aligned} \qquad \text{(5 pts)}$$

(2) Notice that

$$\int \frac{\sin(x)}{\cos^3(x)}\ dx = \int \tan(x)\sec^2(x)\ dx.$$

Then let

$$\begin{aligned}
u &= \tan(x) \\
du &= \sec^2(x)\ dx.
\end{aligned} \qquad \text{(5 pts)}$$

It follows that

$$\begin{aligned}
\int \tan(x)\sec^2(x)\ dx &= \int u\ du \\
&= \frac{u^2}{2} + C \\
&= \frac{\tan^2(x)}{2} + C.
\end{aligned} \qquad \text{(5 pts)}$$

(3) In the previous solution we could instead use the substitution:

$$\begin{aligned}
u &= \sec(x) \\
du &= \sec(x)\tan(x)\ dx.
\end{aligned} \qquad \text{(5 pts)}$$

Then

$$\begin{aligned}
\int \tan(x)\sec^2(x)\ dx &= \int u\ du \\
&= \frac{u^2}{2} + C \\
&= \frac{\sec^2(x)}{2} + C.
\end{aligned} \qquad \text{(5 pts)}$$

Finally we point out that since $\sec^2(x) = \tan^2(x) + 1$, these three answers are all the same (up to a constant).