Math 141 Midterm 3 Question 3 Solution

November 10, 2014

Question: a)

$$\int \frac{x^2}{x^2-1}dx;$$

b)

$$\int \frac{x^3+3x}{(x^2+2)^2}dx.$$

Solution:
a) Rewrite the numerator as x^2-1+1 (or use polynomial long division) and then factor the integrand into

$$\int 1+\frac{1}{x^2-1}dx \qquad \text{4 pts}$$

Split the second term using partial fractions. $\frac{1}{x^2-1} = \frac{A}{x+1} + \frac{B}{x-1}$ and so $1 = A(x-1)+B(x+1) = (A+B)x+(B-A)$. Equate coefficients so $A+B=0$ and $B-A=1$. Solve this system so that

$$A = -\tfrac{1}{2}, B = \tfrac{1}{2} \qquad \text{2 pts}$$

Now integrate $\int 1 - \frac{1}{2}\frac{1}{x+1} + \frac{1}{2}\frac{1}{x-1}dx$ to get

$$x - \tfrac{1}{2}\ln(x+1) + \tfrac{1}{2}\ln(x-1) + C \qquad \text{4 pts}$$

b) Since the degree in the numerator is smaller than the degree of the polynomial in the denominator we can immediately use partial fractions. Note that there is a method of solving this just by using a u-substitution. We have

$$\frac{x^3+3x}{(x^2+2)^2} = \frac{Ax+B}{x^+2} + \frac{Cx+D}{(x^+2)^2} \qquad \text{4 pts}$$

Thus $x^3+3x = (Ax+B)(x^2+2)+Cx+D = Ax^3+2Ax+Bx^2+2B+Cx+D$ and so $x^3+3x = Ax^3+Bx^2+(2A+C)x+(2B+D)$. Equate coefficients so

$$A = 1$$
$$B = 0$$
$$2A + C = 3$$
$$2B + D = 0$$

Thus

$$A = 1, B = 0, C = 1, D = 0 \qquad \text{3 pts}$$

Integrate $\int \frac{x}{x^2+2} + \frac{x}{(x^2+2)^2}dx$. The same u=substitution $u = x^2+2$ with $du = 2xdx$ works for both parts and so

$$\int \tfrac{1}{2}\frac{1}{u} + \tfrac{1}{2}\frac{1}{u^2}du$$

$$= \tfrac{1}{2}\ln(u) - \tfrac{1}{2}\frac{1}{u} + C$$

So finally,

$$= \tfrac{1}{2}\ln(x^2+2) - \tfrac{1}{2}\frac{1}{x^2+2} + C \qquad \text{8 pts}$$