Solution: Exam 1, Problem 4

10th December 2013

Find the Taylor Expansion.

$$f(x) = \frac{x^2}{1+x}$$

Grading scheme :

Shorter method.

$$\frac{1}{1+x} = 1 - x + x^2 - \ldots = \sum_{n=0}^{\infty}(-1)^n x^n \ldots \text{ 10 points}$$

So,

$$\frac{x^2}{1+x} = x^2(1 - x + x^2 - \ldots) = \sum_{n=0}^{\infty}(-1)^n x^{n+2} \ldots \text{ 10 points}$$

If a function equals its power series, then this series is the Taylor series of the function. So, the Taylor series of the given function about $x = 0$ is $\sum_{n=0}^{\infty}(-1)^n x^{n+2} \ldots$ 5 points

Note: No one in the class who has used this method has written this above statement, so everyone has lost 5 points. I have indicated this on the answer papers by mentioning 'statement?'.

Longer method.

Some of you have used the Taylor series formula to differentiate the function $\frac{x^2}{1+x}$.

$$g(x) = \frac{x^2}{1+x} = (x^2)(1+x)^{-1}$$

$$g(0) = 1$$

$$g'(x) = \frac{2x(1+x) - x^2}{(1+x)^2} = \frac{x^2+2x}{(1+x)^2} = \frac{x^2+2x+1-1}{(1+x)^2} = \frac{(1+x)^2-1}{(1+x)^2} = 1 - \frac{1}{(1+x)^2}$$

Hence, $g'(0) = 1 - 1 = 0$

$$g''(x) = (-1)(-2)\frac{1}{(1+x)^3} \text{ Hence, } g''(0) = (-1)^2 2! = 2 \ldots \text{ 7 points}$$

In general, for $k \geq 2$

$$g^{(k)}(x) = (-1)(-2)\ldots(-k)\frac{1}{(1+x)^{k+1}}$$

$$\text{Thus } g^{(k)}(0) = (-1)^k k! \ldots \text{ 11 points}$$

Note $g(0) = 0, g'(0) = 0$. Hence, the Taylor series of $g(x) = \frac{x^2}{1+x}$ around $x = 0$ is,

$$\sum_{n=0}^{\infty} g^{(k)}(0)\frac{x^k}{k!} = \sum_{n=2}^{\infty} g^{(k)}(0)\frac{x^k}{k!} = \sum_{n=2}^{\infty}(-1)^k k!\frac{x^k}{k!} = \sum_{n=2}^{\infty}(-1)^k x^k \ldots \text{ 7 points}$$