Geometric Analysis: Lectures 1-4

1 Basics of Differentiable Manifolds

A differentiable manifold M is a topological space which is locally homeomorphic to an open subset of $\mathbb{R}^n$ and whose transition functions are smooth maps between these open subsets. Using the transition functions we can give coordinates around a point in the manifold. These coordinates will usually be denoted $\{x_1(\cdot), x_2(\cdot), \ldots, x_n(\cdot)\}$ so that $x(p) = (0, \ldots, 0)$.

The tangent space at each point $p \in M$, denoted T_pM, is an n-dimensional vector space spanned by the vectors $\{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\}$. The inner product on this tangent space is given by the metric g at the point p. For any vectors $X, Y \in T_pM$ the inner product is denoted $g(X, Y, p)$. For any point q in a neighborhood of p the metric can be written as a matrix with coefficients $g_{ij}(q) = g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}, q)$. We require the functions g_{ij} to vary smoothly in the coordinate neighborhood on which they're defined.

The metric can be used to measure the length of differentiable curves $\gamma : [a, b] \to M$ with velocity vector fields $\dot{\gamma}(t) \in T_{\gamma(t)}M$

$$L(\gamma) = \int_a^b \sqrt{g(\dot{\gamma}(t), \dot{\gamma}(t), \gamma(t))}\, dt$$

A natural question to ask is what curves minimize length between fixed endpoints $\gamma(a)$, $\gamma(b)$ in (M, g).

2 The Euler-Lagrange Equation of the Length Functional

For a given differentiable manifold the tangent bundle TM is a manifold whose points are (p, X) where $X \in T_pM$. If M is n-dimensional TM is 2n-dimensional, accounting for both the dimesion of M and T_pM. A basic example is for the circle $M = S^1$ which has $T_pM = \mathbb{R}$ at every point, and $TM = S^1 \times \mathbb{R}$.

We define a Lagrangian on M to be a smooth function on the tangent bundle TM.

Thus $L(\gamma)$ is a Lagrangian as it only depends on the points $(\gamma(t), \dot{\gamma}(t)) \in TM$. We now seek to find the critical points of the length functional L by computing its Euler-Lagrange Equation.

Let $\gamma_\epsilon : [a, b] \to M$ be a one-parameter family of curves with fixed endpoints $\gamma_\epsilon(a) = \gamma(a)$, $\gamma_\epsilon(b) = \gamma(b)$, $\gamma_0 = \gamma$, and $\gamma_\epsilon(t)$ C^∞ in ϵ and t.

If γ is a critical point of the length function L then for any one-parameter family γ_ϵ, $\frac{d}{dt}\big|_{\epsilon=0} L(\gamma_\epsilon) = 0$.

To compute the above derivative and derive the Euler-Lagrange equation we first define coordinates $\{x_1, \ldots, x_n\}$ on an open neighborhood $U_p \subset M$ of a point p and coordinates $\{p_1, \ldots, p_n\}$ on the tangent space so that for an arbitrary Lagrangian $\int F(\gamma, \dot{\gamma})$ we can compute

$$\frac{d}{d\epsilon}\bigg|_{\epsilon=0} \int F(\gamma_\epsilon, \dot{\gamma}_\epsilon)\, dt = \int_a^b \left(\sum_i \frac{\partial F}{\partial x_i} \frac{\partial \gamma_\epsilon^i}{\partial \epsilon}\bigg|_{\epsilon=0} + \sum_i \frac{\partial F}{\partial p_i} \frac{\partial^2 \gamma_\epsilon^i}{\partial \epsilon \partial t}\bigg|_{\epsilon=0} \right) dt$$

After integrating by parts

$$= \int_a^b \sum_i \frac{\partial \gamma^i_\epsilon}{\partial \epsilon}\Big|_{\epsilon=0} \left(\frac{\partial F}{\partial x_i} - \frac{d}{dt}\frac{\partial F}{\partial p_i} \right) dt$$

Thus $\frac{d}{d\epsilon}\big|_{\epsilon=0} F(\gamma_\epsilon) = 0$ imples

$$\sum_i \frac{\partial F}{\partial x_i} - \frac{d}{dt}\frac{\partial F}{\partial p_i} = 0$$

which is the Euler-Lagrange equation of $\int F(\dot{\gamma}, \gamma)$. Now we compute the Euler-Lagrange equation for the specific case of the energy functional E which has the same critical points as the length functional L.

$$E(\gamma, \dot{\gamma}) = g_{kl}\dot{\gamma}^k(t)\dot{\gamma}^l(t)$$

suppressing the summation over repeated indices. The Euler-Lagrange equation becomes:

$$\begin{aligned}
\sum_i \frac{\partial g_{kl}}{\partial x_i}\dot{\gamma}^k\dot{\gamma}^l &= \frac{d}{dt}\left(\sum_i g_{kl}\dot{\gamma}^k \delta^l_i + \sum_i g_{kl}\dot{\gamma}^l \delta^k_i \right) \\
&= \frac{d}{dt}\left(\sum_i g_{il}\dot{\gamma}^l + \sum_i g_{ki}\dot{\gamma}^k \right) \\
&= \sum_i g_{il}\ddot{\gamma}^l + g_{ki}\ddot{\gamma}^k + \frac{\partial g_{il}}{\partial x_m}\dot{\gamma}^m\dot{\gamma}^l + \frac{\partial g_{ki}}{\partial x_m}\dot{\gamma}^m\dot{\gamma}^k
\end{aligned}$$

By using the symmetry that $g_{ij} = g_{ji}$, reindexing and combining the $\ddot{\gamma}$ terms, and rewriting $\frac{\partial g_{ij}}{\partial x_k} = g_{ij,k}$ we get

$$2\sum g_{il}\ddot{\gamma}^l = \sum \left(g_{mk,i} - g_{im,k} - g_{ik,m} \right) \dot{\gamma}^k\dot{\gamma}^m$$

Thus a curve γ is a critical point of the energy functional E, and thus a critical point of the length functional L, if it is a solution to this differential equation. Next we will show that this is the same geodesic equation that arises in differential geometry.

3 Geodesics from a Riemannian Geometry Viewpoint

Let (M, g) be a Riemannian manifold. The geodesic equation can also be derived by considering what it means for a curve to have zero acceleration in a manifold. The vector field $\dot{\gamma}$ is a well defined first derivative for a curve γ, but in order to have a well defined second derivative we must define a connection on M which will allow us to differentiate one vector field along another. An affine connection on M is a map $\nabla : TM \times TM \to TM$ satisfying two conditions.

$$\begin{aligned}
&1)\nabla_{aX+bY}Z = a\nabla_X Z + b\nabla_Y Z \\
&2)\nabla_X(aY) = a\nabla_X Y + da(X)Y = a\nabla_X Y + X(a)Y
\end{aligned}$$

for X, Y, Z sections of TM and a, b smooth functions on M

The Levi-Civita connection or Riemannian connection is an affine connection with the added properties of being symmetric and compatible with the metric.

$$3)\nabla_X Y - \nabla_Y X = [X,Y]$$
$$4)Xg(X,Y) = g(\nabla_X Y, X) + g(Y, \nabla_X Z)$$

With the Levi-Civita connection we can define a vector field X to be parallel along a curve γ if it satisfies $\nabla_{\dot{\gamma}(t)}X(t) = 0$. Because of the compatibility of ∇ with the metric g, the inner product $g(X(t), Y(t))$ of two vector fields which are parallel along γ is constant.

In order to facilitate computations with the connection we define the action of ∇ on the basis for the tangent space in local coordinates $\{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\}$ which we will furthermore denote $\{\partial_1, \ldots, \partial_n\}$.

$$\nabla_{\partial_i}\partial_j = \Gamma^k_{ij}\partial_k$$
$$\Gamma_{ijk} = g_{kl}\Gamma^l_{ij}$$

Γ^k_{ij} are called Christoffel symbols of the 2^{nd} kind and Γ_{ijk} are called Christoffel symbols of the 1^{st} kind.

Lemma 3.1. *The Levi-Civita connection is unique.*

Proof. We will use the metric compatibility and apply this to basis vector fields.

$$\begin{aligned}
\partial_k g(\partial_i, \partial_j) &= g_{ij,k} \\
&= g(\nabla_{\partial_k}\partial_i, \partial_j) + g(\partial_i, \nabla_{\partial_k}\partial_j) \\
&= g(\Gamma^l_{ki}\partial_l, \partial_j) + g(\partial_i, \Gamma^l_{kj}\partial_l) \\
&= \Gamma^l_{ki}g_{lj} + \Gamma^l_{kj}g_{il} \\
&= \Gamma_{kij} + \Gamma_{kji}
\end{aligned}$$

By the symmetry of the connection

$$\begin{aligned}
0 &= [\partial_i, \partial_j] \\
&= \nabla_{\partial_i}, \partial_j - \nabla_{\partial_j}\partial_i \\
&= (\Gamma^k_{ij} - \Gamma^k_{ji})\partial_k
\end{aligned}$$

$\implies \Gamma^k_{ij} = \Gamma^k_{ji}$
and summing covariant derivatives of the metric

$$\begin{aligned}
g_{ij,k} + g_{jk,i} - g_{ki,j} &= \Gamma_{kij} + \Gamma_{kji} + \Gamma_{ijk} + \Gamma_{ikj} - \Gamma_{jki} - \Gamma_{jik} \\
&= 2\Gamma_{ijk}
\end{aligned}$$

So the Christoffel symbols are determined by the metric and the connection is unique. □

Now we define geodesics from a Riemannian geometry viewpoint as curves $\gamma : [a,b] \to M$ which have zero acceleration, or equivalently, parallel velocity vector fields $\dot{\gamma}$. Thus, the geodesic equation becomes $\nabla_{\dot{\gamma}}\dot{\gamma}$. In local coordinates, written in terms of the Christoffel symbols of the Riemmanian connection this becomes:

$$\gamma(t) = (x^1(t), \ldots, x^n(t))$$
$$\dot{\gamma}(t) = (\dot{x}^1(t), \ldots, \dot{x}^n(t))$$

$$
\begin{aligned}
\nabla_{\dot\gamma}\dot\gamma &= \nabla_{\dot x^i \partial_i}\dot x^j \partial_j \\
&= \dot x^i \dot x^j \Gamma^k_{ij}\partial_k + \ddot x^k \partial_k \\
&= (\dot x^i \dot x^j \Gamma^k_{ij} + \ddot x^k)\partial_k = 0
\end{aligned}
$$

So the geodesic equation is

$$\dot\gamma^i \dot\gamma^j \Gamma^k_{ij} + \ddot\gamma^k = 0$$

To show this is the same as the geodesic equation which we arrived at by variational methods

$$
\begin{aligned}
0 &= 2\dot\gamma^i \dot\gamma^j \Gamma^k_{ij} + 2\ddot\gamma^k \\
&= 2g_{lk}\ddot\gamma^k + 2g_{lk}\Gamma^k_{ij}\dot\gamma^i\dot\gamma^j \\
&= 2g_{lk}\ddot\gamma^k + 2\Gamma_{ijl}\dot\gamma^i\dot\gamma^l \\
&= 2g_{lk}\ddot\gamma^k - (g_{li,j} - g_{ij,l} - g_{jl,i})\dot\gamma^i\dot\gamma^j
\end{aligned}
$$

which agrees with our variational method up to a change of indices.

4 Jacobi's Theorem

Next we seek to know when a critical point of the length functional is in fact a minimizer. All geodesics exist and are length minimizing in a small ball around any point. We describe these geodesics by the exponential map. Let γ be a geodesic in M such that $\gamma(0) = p$, $\dot\gamma(0) = X$. Then γ has constant speed $\sqrt{g(X,X)}$. The exponential map $exp_p : T_pM \to M$ is defined:

$$exp_p(X) = \gamma(1) \qquad\qquad exp_p(tX) = \gamma(t)$$

To find when a geodesic is length minimizing, we must understand the critical points of exp_p, so we need to know when $dexp_p$ has a nontrivial kernel. Thus we define a point $q = exp_p(tA)$ of a geodesic γ to be conjugate to p along γ if it is a critical value of exp_p. With this definition we can state the main theorem regarding length minimizing geodesics.

Theorem 4.1. *If γ is a minimizing geodesic then none of its interior points are conjugate points.*

Before we prove this theorem we must relate conjugate points to variations of a geodesic. Let $\gamma_\epsilon(t)$ be a variation such that $\gamma_0(t) = \gamma(t)$, $\gamma_\epsilon(0) = \gamma(0)$, and $\gamma_\epsilon(1) = \gamma(1)$. Our strategy is to assume the existence of a conjugate point along γ and show that this implies the existence of a variation γ_ϵ such that $\frac{\partial^2}{\partial\epsilon^2}L(\gamma_\epsilon) < 0$. Such a variation would show γ to in fact be a locally maximizing geodesic, thus proving the theorem.

$$
\begin{aligned}
\frac{\partial}{\partial\epsilon}L(\gamma_\epsilon) &= \int_0^1 \frac{1}{2|\dot\gamma|}\frac{\partial}{\partial\epsilon}\left\langle \dot\gamma, \dot\gamma\right\rangle\, dt \\
&= \int_0^1 \frac{1}{2|\dot\gamma|} 2\left\langle \nabla_{\frac{\partial\gamma}{\partial\epsilon}}\dot\gamma, \dot\gamma\right\rangle\, dt
\end{aligned}
$$

$$\frac{\partial^2}{\partial\epsilon^2}L(\gamma_\epsilon) = \int_0^1 \frac{-1}{|\dot\gamma|^2}\frac{1}{|\dot\gamma|}\left\langle \nabla_{\frac{\partial\gamma}{\partial\epsilon}}\dot\gamma, \dot\gamma\right\rangle^2 + \frac{1}{|\dot\gamma|}\left[\left\langle \nabla_{\frac{\partial\gamma}{\partial\epsilon}}\nabla_{\frac{\partial\gamma}{\partial\epsilon}}\dot\gamma, \dot\gamma\right\rangle + \left|\nabla_{\frac{\partial\gamma}{\partial\epsilon}}\dot\gamma\right|^2\right] dt$$

Now call $\frac{\partial \gamma}{\partial \epsilon}\Big|_{\epsilon=0} = \gamma' = X$ so that $X(0) = 0$ and $X(1) = 0$.

Since $\frac{\partial}{\partial \epsilon}$ and $\frac{\partial}{\partial t}$ are commuting vector fields in $\mathbb{R}^n$ their pushforwards commute as $\left[\gamma_* \frac{\partial}{\partial \epsilon}, \gamma_* \frac{\partial}{\partial t}\right] = \gamma_* \left[\frac{\partial}{\partial \epsilon}, \frac{\partial}{\partial t}\right] = 0$.

This implies $0 = [\dot{\gamma}, X] = \nabla_{\dot{\gamma}} X - \nabla_X \dot{\gamma}$.

Thus, the first term in the integral of $\frac{\partial^2}{\partial \epsilon^2}\Big|_{\epsilon=0} L(\gamma_\epsilon)$ is

$$\frac{-1}{|\dot{\gamma}|^3} \langle \nabla_{\dot{\gamma}} X, \dot{\gamma} \rangle^2 = \frac{-1}{|\dot{\gamma}|^3} \left(\frac{d}{dt} \langle X, \dot{\gamma} \rangle \right)^2$$

To rewrite the second term of the integral we define the Riemannian curvature tensor as a map $R : TM \times TM \to TM$ defined as

$$R(X,Y)Z = \left(\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}\right) Z$$

which we can rewrite as a map $R : TM \times TM \times TM \times TM \to \mathbb{R}$ defined as

$$R(X,Y,Z,W) = g(R(X,Y)Z, W)$$

Now the second term of the integral can be rewritten as

$$\begin{aligned} \frac{1}{|\dot{\gamma}|} \left[\langle \nabla_X \nabla_{\dot{\gamma}} X, \dot{\gamma} \rangle + |\nabla_X \dot{\gamma}|^2 \right] &= \frac{1}{|\dot{\gamma}|} \left[- \langle \nabla_{\dot{\gamma}} \nabla_X X, \dot{\gamma} \rangle + \langle R(X,\dot{\gamma})X, \dot{\gamma} \rangle + |\nabla_X \dot{\gamma}|^2 \right] \\ &= -\frac{d}{dt} \langle \nabla_X X, \dot{\gamma} \rangle + \langle R(X,\dot{\gamma})X, \dot{\gamma} \rangle + |\nabla_X \dot{\gamma}|^2 \end{aligned}$$

And if we rescale γ to have constant unit speed the second variational formula for the length functional becomes

$$\frac{\partial^2}{\partial \epsilon^2}\Big|_{\epsilon=0} L(\gamma_\epsilon) = \int_a^b - \left(\frac{d}{dt} \langle X, \dot{\gamma} \rangle \right)^2 + \langle R(X,\dot{\gamma}), X, \dot{\gamma} \rangle + |\nabla_X \dot{\gamma}|^2 \, dt$$

Now we consider normal fields X such that $\langle X(t), \dot{\gamma}(t) \rangle = 0$ for any t. If X is normal along γ then the second variational formula becomes

$$I_\gamma(X) = \frac{\partial^2}{\partial \epsilon^2}\Big|_{\epsilon=0} L(\gamma_\epsilon) = \int_a^b R(X,\dot{\gamma},X,\dot{\gamma}) + |\nabla_X \dot{\gamma}|^2 \, dt$$

Now we will seek minimizers of I and prove they have negative values when γ contains conjugate points. We consider I_γ over all vector fields along γ, which are sections of $\gamma^* TM$, vanishing at 0 and 1. $\gamma^* TM \simeq [0,1] \times \mathbb{R}^n$ and we can make this explicit by choosing a frame for $T_p M$ which varies along γ. First choose $\{E_1, \ldots, E_n\}$, an orthonormal frame for $T_{\gamma(0)} M$ and extend $E_i(t) \in T_{\gamma(t)} M$ by the solution to $\nabla_{\dot{\gamma}} E_i(t) = 0$, $E_i(0) = E_i$. It can be shown that $E_i(t)$ is an orthonormal frame at each point of γ.

Decomposing a vector field X along γ as $X = \sum q_i(t) E_i(t)$, and $\dot{\gamma}(t) = \sum a_i E_i(t)$. Then $\nabla_{\dot{\gamma}} X = \sum \dot{q}_i^2(t) E_i(t)$.

Now the second variational formula for X along γ is

$$I_\gamma X = \int_a^b \sum \dot{q}_i^2 + R_{ijkl} \, q_i a_j q_k a_l \, dt$$

where $R_{ijkl} = R(E_i, E_j, E_k, E_l)$.

Using the Euler-Lagrange equation for functionals of this type that we derived previously we find that the Euler-Lagrange equation for I is

$$2\ddot{q}_i = -R_{ijkl}\, a_j q_k a_l - R_{mjil}\, q_m a_j a_l$$

And using that $\sum \ddot{q}_i E_i = \nabla_{\dot{\gamma}} \nabla_{\dot{\gamma}} X$ we get the Jacobi Equation

$$\nabla_{\dot{\gamma}} \nabla_{\dot{\gamma}} X + R(\dot{\gamma}, X)\dot{\gamma} = 0$$

When X satisfies this equation it is called a Jacobi field. We seek to find a normal Jacobi field X with $I(X) < 0$ along a geodesic γ with a conjugate point, which will provide the contradiction needed to prove Theorem 4.1. But first we must show that any Jacobi field X with $X(0) = 0$ and $X(1) = 0$ can be obtained from a variation so that showing $I(X) < 0$ is sufficient for proving the theorem which concerns variations rather than Jacobi fields.

Proposition 4.2. *There is a bijection between variations $\gamma_\epsilon(t)$ which are composed of geodesics and have a fixed start point $\gamma(0)$ and Jacobi fields $X(t)$ with $X(0) = 0$.*

Proof. First we'll show that a variation of geodesics with a fixed start point induces a Jacobi field X with $X(0) = 0$. Such a variation can be defined as $\gamma_\epsilon(t) = exp_p tA(\epsilon)$ with $A(\epsilon) \in T_pM$ and $A(0) = \dot{\gamma}(0)$. The induced Jacobi field is $X(t) = \gamma'_\epsilon(t) = dexp_p|_{tA(\epsilon)}\,(tA'(0))$. Clearly $X(0) = 0$.

Now we'll show that $\gamma'_\epsilon(t)$ is a Jacobi field.

$$\begin{aligned}
&\nabla_{\dot{\gamma}} \nabla_{\dot{\gamma}} \gamma' + R(\gamma', \dot{\gamma})\dot{\gamma} = 0 \\
&\iff \nabla_{\dot{\gamma}} \nabla_{\gamma'} \dot{\gamma} + R(\gamma', \dot{\gamma})\dot{\gamma} = 0 \\
&\iff \nabla_{\gamma'} \nabla_{\dot{\gamma}} \dot{\gamma} + R(\dot{\gamma}, \gamma')\dot{\gamma} + R(\gamma', \dot{\gamma})\dot{\gamma} = 0 \\
&\iff \nabla_{\gamma'} \nabla_{\dot{\gamma}} \dot{\gamma} = 0
\end{aligned}$$

But the last line being equal to 0 is exactly the geodesic equation, so γ'_ϵ is a Jacobi field.

Next we'll show that a Jacobi field X with $X(0) = 0$ gives rise to a variation of geodesics with a fixed start point. But first we'll give an argument for why for each variation with fixed start point $\phi(t, \epsilon) = \gamma_\epsilon(t) = exp_p(tA(\epsilon))$ which gives rise to a variational field $X(t) = \frac{\partial \phi}{\partial \epsilon}(t, 0)$ has its covariant derivative $\nabla_{\dot{\gamma}} X(0) = A'(0)$. This relies on the fact that

$$\nabla_{\dot{\gamma}} X(t) = \nabla_{\dot{\gamma}} \left(\left.\frac{\partial \phi}{\partial \epsilon}\right|_{\epsilon=0} \right) = \left.(\nabla_{\gamma'} \dot{\gamma}_\epsilon(t))\right|_{\epsilon=0}$$

Where in the last equality I used the relation $\nabla_{\dot{\gamma}} \frac{\partial \phi}{\partial \epsilon} = \nabla_{\gamma'} \frac{\partial \phi}{\partial t}$ between partial and covariant derivatives on the surface which is the image of ϕ. To see why this relation is true choose a local coordinate x for M near a point on the image of ϕ and compute

$$\begin{aligned}
\nabla_{\dot{\gamma}} \left(\frac{\partial \phi}{\partial \epsilon} \right) &= \nabla_{\dot{\gamma}} \left(\frac{\partial x^i}{\partial \epsilon} \frac{\partial}{\partial x^i} \right) \\
&= \frac{\partial^2 x^i}{\partial t \partial \epsilon} \frac{\partial}{\partial x^i} + \frac{\partial x^i}{\partial \epsilon} \nabla_{\frac{\partial x^j}{\partial t} \frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i} \\
&= \frac{\partial^2 x^i}{\partial t \partial \epsilon} \frac{\partial}{\partial x^i} + \frac{\partial x^i}{\partial \epsilon} \frac{\partial x^j}{\partial t} \nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i}
\end{aligned}$$

And the symmetry of the connection gives $\nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i} = \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j}$ so a computation of $\nabla_{\gamma'} \frac{\partial \phi}{\partial t}$ gives the same result.

Thus $\nabla_{\dot\gamma}X(t) = (\nabla_{\gamma'}\dot\gamma_\epsilon(t))|_{\epsilon=0}$ and $\nabla_{\dot\gamma}X(0) = (\nabla_{\gamma'}A(\epsilon))|_{\epsilon=0}$. But since at $t=0$ the the curve $\epsilon \mapsto \gamma_\epsilon(0)$ is constant, the covariant derivative is just a partial derivative with respect to ϵ and $\nabla_{\dot\gamma}X(0) = A'(0)$.

Now that we've established that $\nabla_{\dot\gamma}X(0) = A'(0)$, we continue the proof by considering an arbitrary Jacobi field with initial value 0.

Given such an X let $B = \nabla_{\dot\gamma}X(0)$. Now define $A(\epsilon) = \dot\gamma(0) + \epsilon B$ and consider the variation $\gamma_\epsilon(t) = exp_p tA(\epsilon)$. The induced Jacobi field is $Y(t) = \gamma'_\epsilon(t) = dexp_p|_{tA(\epsilon)}(tB)$. But $Y(0) = X(0) = 0$ and $\nabla_{\dot\gamma}Y(0) = B = \nabla_{\dot\gamma}X(0)$ by the argument given above, so by the uniqueness of ordinary differential equations $X(t) = Y(t)$ and $X(t)$ is induced by a variation of geodesics with a fixed start point. □

Before proving the existence of a Jacobi field with $I(X) < 0$, we will first prove a helpful proposition.

Proposition 4.3. *If X is a Jacobi field such that $\langle X(0), \dot\gamma(0)\rangle = \langle X(1), \dot\gamma(1)\rangle = 0$ then X is a normal Jacobi field, and if furthermore $X(0) = X(1) = 0$ then $I(X) = 0$.*

Proof. To show that X is normal we must show $\langle \dot\gamma, X\rangle = 0$. First we note

$$\frac{d}{dt}\langle \dot\gamma, X\rangle = \langle \dot\gamma, \nabla_{\dot\gamma}X\rangle$$

And taking a further derivative

$$\begin{aligned}\frac{d^2}{dt^2}\langle \dot\gamma, \nabla_{\dot\gamma}X\rangle &= \langle \dot\gamma, \nabla_{\dot\gamma}\nabla_{\dot\gamma}X\rangle \\ &= -\langle \dot\gamma, R(X,\dot\gamma)\dot\gamma\rangle \\ &= -R(\dot\gamma,\dot\gamma,X,\dot\gamma) = 0\end{aligned}$$

So $\langle \dot\gamma, X\rangle = at + b$ for $a, b \in \mathbb{R}$ and by using the conditions on the endpoints we can show that $\langle \dot\gamma, X\rangle = 0$, so X is normal. Thus, the equation for $I(\gamma)$ reduces to:

$$\begin{aligned}I(X) &= \int_0^1 |\nabla_{\dot\gamma}X|^2 - R(X,\dot\gamma,X,\dot\gamma)\,dt \\ &= \int_0^1 \langle \nabla_{\dot\gamma}X, \nabla_{\dot\gamma}X\rangle + \langle X, \nabla_{\dot\gamma}\nabla_{\dot\gamma}X\rangle\ dt \\ &= \int_0^1 \frac{d}{dt}\langle X, \nabla_{\dot\gamma}X\rangle\ dt \\ &= \langle X, \nabla_{\dot\gamma}X\rangle|_0^1 = 0\end{aligned}$$

as the conditions on the endpoints are the same. □

Now we have all of the tools necessary to prove Theorem 4.1 which states that if γ is a minimizing geodesic then none of its interior points are conjugate points.

Proof. We will prove the theorem by contradiction, assuming there exists a conjugate point at $\gamma(t_0)$ and showing that γ cannot be length minimizing. To this end we will show there exists a normal Jacobi field Y along γ, which by Proposition 4.2 is the variational field of some variation $\gamma_\epsilon(t)$, such that the second variation of Y, $I(Y) < 0$ which would imply γ is locally length maximizing.

First we show the existence of a normal Jacobi field such that $X(0) = 0$ and $X(t_0) = 0$, where $\gamma(t_0)$ is the conjugate point to $\gamma(0)$. Since $\gamma(t_0)$ is conjugate, there exists $A(\epsilon) \in T_pM$ such that $A'(0) \in ker\left(dexp_p|_{t_0A}\right)$, and defining $X(t) = dexp_p|_{tA(\epsilon)}(tA'(t))$ gives the desired Jacobi field. And by Proposition 4.3 it is normal on $[0, t_0]$ and has $I(X) = 0$ on that interval.

Around every point in a Riemannian manifold the exponential map is a diffeomorphism from a small ball in the tangent space to a small normal neighborhood in the manifold. Thus there exists $\delta > 0$ such that the exponential map is a diffeomorphism on $B(0, \delta) \subset T_{\gamma(t_0)}M$. Thus $exp_{\gamma(t_0)}$ cannot have any critical points in the interval, so none of the points on $\gamma|_{[t_0-\delta,t_0+\delta]}$ are conjugate to $\gamma(t_0 - \delta)$. Now we seek to prove the existence of a Jacobi field $Z(t)$ on $[t_0 - \delta, t_o + \delta]$ such that $Z(t_0 - \delta) = X(t_0\delta)$ and $Z(t_0 + \delta) = 0$.

We first note that since the Jacobi equation is an system of n second-order ordinary differential equations, its solution set is 2n-dimensional. Next, we consider the linear map $\alpha : X \mapsto (X(t_0 - \delta), X(t_0 + \delta))$ from the 2n-dimensional space of Jacobi fields to the 2n-dimensional space $T_{\gamma(t_0-\delta)}M \oplus T_{\gamma(t_0+\delta)}M$. It is injective as it is a homomorphism of vector spaces and the only Jacobi field which has both endpoints 0 is the unique solution $X(t) = 0$ of the Jacobi equation with those initial conditions. But since the dimension of the domain and codomain are both 2n it is surjective as well. Thus there exists a Jacobi field Z with the perscribed endpoints. By *Proposition 4.3* Z is normal.

Before we define our final Jacobi field which will satisfy $I(Y) < 0$, we will first show that the Jacobi field Z defined above is minimal in the sense that $I(Z) < I(W)$ where W is any other piecewise smooth field along γ. Since Z is normal, it can be written as an $\mathbb{R}$-linear combination of of $\{W_1, \ldots, W_{n-1}\}$, a basis for the subspace of normal Jacobi fields. A computation shows that for a smooth field $W = f^i W_i$, where now f^i are smooth functions, I can be written.

$$I(W) = \int_0^1 \left| \dot{f}^i W_i \right|^2 dt + f^i(1) f^j(1) \left. \langle \nabla_{\dot{\gamma}} W_i, W_j \rangle \right|_{t=1}$$

And since Z is normal, the f^is are constant, and the formula reduces to.

$$I(Z) = f^i(1) f^j(1) \left. \langle \nabla_{\dot{\gamma}} W_i, W_j \rangle \right|_{t=t_0+\delta}$$

Thus for any smooth field W which agrees with Z on $t_0 - \delta$ and $t_0 + \delta$ we have

$$I(W) - I(Z) = \int_{t_0-\delta}^{t_0+\delta} \left| \dot{f}^i W_i \right|^2 dt \geq 0$$

And so Z is strictly minimal amongst smooth fields with the same endpoints.

Now define the Jacobi field Y on all of γ by

$$Y(t) = \begin{cases} X(t) & t \in [0, t_0 - \delta] \\ Z(t) & t \in [t_0 - \delta, t_0 + \delta] \\ 0 & t \in [t_0 + \delta, 1] \end{cases}$$

And we compute

$$\begin{aligned} I_0^1(Y) = I_0^{t_0+\delta}(Y) &= I_0^{t_0-\delta}(X) + I_{t_0-\delta}^{t_0+\delta}(Z) \\ &= I_0^{t_0}(X) - I_{t_0-\delta}^{t_0}(X) + I_{t_0\delta}^{t_0+\delta}(Z) \\ &= I_{t_0-\delta}^{t_0+\delta}(Z) - I_{t_0-\delta}^{t_0} \end{aligned}$$

And Z is minimal by the argument above, so for a field defined

$$W(t) = \begin{cases} X(t) & t \in [t_0 - \delta, t_0] \\ 0 & t \in [t_0, t_0 + \delta] \end{cases}$$

W and Z have the same endpoints, so $I(Z) < I(W)$, and since $I_{t_0-\delta,t_0}^{t_0}(X) = I_{t_0-\delta}^{t_0+\delta}(W)$, it follows that $I_0^1(Y) < 0$, contradicting the fact that γ was length minimizing.

□

Geometric Analysis Homework 1

The first four exercises are taken from Geometry VI: Riemannian Geometry by M.M. Postnikov.

Exercise 11.3 Prove that parallel translation with respect to a Riemannian connection is an isometric mapping of tangent spaces.

Proof. Let $X(t)$ and $Y(t)$ be parallel vector fields along γ so that $\nabla_{\dot{\gamma}(t)}X(t) = \nabla_{\dot{\gamma}(t)}Y(t) = 0$. To prove parallel translation is an isometry we must show $\langle X(t), Y(t)\rangle$ is constant.

$$\frac{d}{dt}\langle X(t), Y(t)\rangle = \left\langle \nabla_{\dot{\gamma}(t)}X(t), Y(t)\right\rangle + \left\langle X(t), \nabla_{\dot{\gamma}(t)}Y(t)\right\rangle = 0$$

Thus parallel translation is an isometry. □

Exercise 11.4 Prove that the matrix for the Lagrangian L on TM with local coordinates $\left(q^1, \ldots, q^n, \dot{q}^1, \ldots, \dot{q}^n\right)$ given by

$$\left[\frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}\right]$$

does not depend on the choice of local coordinates.

Proof. Let ϕ be a change of coordinates such that $\phi(q) = p$. This induces a change of coordinates on the tangent space by $\dot{p} = (d\phi)^{-1}\dot{q}$ and more specifically $\dot{p}_i = (d\phi^{-1})_{ji}\dot{q}_j$. Thus $\frac{\partial \dot{p}_i}{\partial \dot{q}^j} = (d\phi^{-1})_{ji}$ and $\frac{\partial^2 \dot{p}_i}{\partial \dot{q}^j \partial \dot{q}^k} = 0$. Under the change of coordinates L changes as

$$L(q, \dot{q}) = L(\phi^{-1}(p), d\phi(\dot{p})) = G(p, \dot{p})$$

And the partial derivatives of L change as

$$\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial G}{\partial \dot{p}_k}\frac{\partial \dot{p}_k}{\partial \dot{q}_i}$$

$$
\begin{aligned}
\frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j} &= \frac{\partial}{\partial \dot{q}^j}\left(\frac{\partial G}{\partial \dot{p}_k}\frac{\partial \dot{p}_k}{\partial \dot{q}_i}\right) \\
&= \frac{\partial G}{\partial \dot{p}_k}\frac{\partial^2 \dot{p}_k}{\partial \dot{q}_i \partial \dot{q}_j} + \frac{\partial^2 G}{\partial \dot{p}_k \partial \dot{p}_l}\frac{\partial \dot{p}_l}{\partial \dot{q}_j}\frac{\partial \dot{p}_k}{\partial \dot{q}_i} \\
&= \frac{\partial^2 G}{\partial \dot{p}_k \partial \dot{p}_l}\frac{\partial \dot{p}_l}{\partial \dot{q}_j}\frac{\partial \dot{p}_k}{\partial \dot{q}_i} \\
&= \frac{\partial^2 L}{\partial \dot{p}_k \partial \dot{p}_l}\left(\frac{\partial \dot{q}_j}{\partial \dot{p}_i}\frac{\partial \dot{q}_i}{\partial \dot{p}_j}\right)\left(\frac{\partial \dot{p}_l}{\partial \dot{q}_j}\frac{\partial \dot{p}_k}{\partial \dot{q}_i}\right) \\
&= \frac{\partial^2 L}{\partial \dot{p}_k \partial \dot{p}_l}\delta_i^l \delta_j^k \\
&= \frac{\partial^2 L}{\partial \dot{p}_j \partial \dot{p}_i}
\end{aligned}
$$

So $\left[\frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}\right]$ does not depend on the choice of local coordinates. ☐

Exercise 12.2 Prove that a geodesic remains a geodesic under a reparameterization iff this reparameterization is linear (has the form $t \mapsto at + b$, where $a \neq 0$).

Proof. Consider a reparameterization $s = g(t)$ for the geodesic $\gamma(s)$. Now put the reparameterized curve $\gamma(g(t))$ into the geodesic equation.

$$
\begin{aligned}
&\frac{d^2}{dt^2}\gamma^i(g(t)) + \Gamma^i_{kj}(\gamma(g(t)))\frac{d}{dt}\gamma^j(t)\frac{d}{dt}\gamma^k(t) \\
&= \ddot{\gamma}^i(g(t))(g'(t))^2 + \dot{\gamma}^i(g(t))g''(t) + \Gamma^i_{kj}(\gamma(g(t))\dot{\gamma}^j(g(t))\dot{\gamma}^k(g(t))(g'(t))^2 \\
&= \dot{\gamma}^i(s)g''(t) + \left[\ddot{\gamma}^i(s) + \Gamma^i_{kj}(\gamma(s)\dot{\gamma}^j(s)\dot{\gamma}^k(s)\right] g''(t) \\
&= \dot{\gamma}^i(s)g''(t)
\end{aligned}
$$

Where the last equality is because $\gamma(s)$ is a geodesic. Since $\dot{\gamma}^i(s) \neq 0$, $\gamma(g(t))$ is a geodesic iff $g''(t) = 0$ iff $g(t) = at + b$ for $a, b \in \mathbb{R}$ and $a \neq 0$. ☐

Exercise 27.1 Show that for any piecewise smooth field X on the geodesic γ, the inequality

$$I_0^1(X^\perp) \leq I_0^1(X)$$

holds, where $X^\perp$ is the normal component of the field X (i.e., $X(t) - X^\perp(t)$ is collinear to the vector $\dot{\gamma}(t)$ for every t).

Proof. Assuming $\gamma : [a, b] \to M$ is a unit speedgeodesic, let $\{\dot{\gamma}(a) = A_1, A_2, \ldots, A_n\}$ be an orthonormal basis for $T_{\gamma(a)}M$. Parallel transporting each of these vectors along γ based on the unique solution to the differential equation $\nabla_{\dot{\gamma}(t)}X_i(t) = 0, X_i(0) = A_i$ yields an orthonormal frame along γ with $X_1(t) = \dot{\gamma}(t)$. Thus, any smooth vector field X along γ can be written as $X = \sum_{i=1}^{n} f^i X_i$ for f^i smooth functions. Then $X^\perp = \sum_{i=2}^{n} f^i X_i$ as $X^\perp$ has no component parallel to $X_1 = \dot{\gamma}$. On smooth vector fields $I(x)$ is given by

$$\int_a^b \left[\langle \nabla_{\dot{\gamma}} X, \nabla_{\dot{\gamma}} X\rangle - R(X, \dot{\gamma}, X, \dot{\gamma}) - \langle \nabla_{\dot{\gamma}} X, \dot{\gamma}\rangle^2\right] dt$$

Examining the first term in the integral we see

$$\begin{aligned}
\langle \nabla_{\dot\gamma} X, \nabla_{\dot\gamma} X \rangle &= \left\langle \nabla_{\dot\gamma} f^i X_i, \nabla_{\dot\gamma} f^j X_j \right\rangle \\
&= \left\langle \dot f^i X_i + f^i \nabla_{\dot\gamma} X_i, \dot f^j X_j + f^j \nabla_{\dot\gamma} X_j \right\rangle \\
&= \sum \left| \dot f^i \right|^2
\end{aligned}$$

And comparing X to $X^\perp$ we see

$$\int_a^b \langle \nabla_{\dot\gamma} X, \nabla_{\dot\gamma} X \rangle \, dt - \int_a^b \left\langle \nabla_{\dot\gamma} X^\perp, \nabla_{\dot\gamma} X^\perp \right\rangle \, dt = \int_a^b \left| \dot f^1 \right|^2 \, dt$$

Examining the second term in the integral, since $R(X, Y, Z, Z) = 0$ by the symmetries of the curvature tensor, it follows that $R(f^i X_i, \dot\gamma, f^1 X_1, \dot\gamma) = R(f^i X_i, \dot\gamma, f^1 X_1, X_1) = 0$. Thus $R(X, \dot\gamma, X, \dot\gamma) = R(X^\perp, \dot\gamma, X^\perp, \dot\gamma)$ and their difference contributes nothing to the integral.

Examining the third term in the integral we see

$$\begin{aligned}
\langle \nabla_{\dot\gamma} X, \dot\gamma \rangle^2 &= \left\langle f^i X_i, X_1 \right\rangle^2 \\
&= \left| f^1 \right|^2
\end{aligned}$$

And comparing X to $X^\perp$ we see

$$- \int_a^b \langle \nabla_{\dot\gamma} X, \dot\gamma \rangle^2 \, dt - \left(- \int_a^b \left\langle \nabla_{\dot\gamma} X^\perp, \dot\gamma \right\rangle^2 \, dt \right) = - \left| f^1 \right|^2$$

So together

$$I(X) - I(X^\perp) = \int_a^b \left| f^1 \right|^2 - \left| f^1 \right|^2 \, dt = 0$$

□

Exercise 5 Prove that the Jacobi field Z from the proof of *Theorem 4.1* from the notes on Lectures 1-4 is normal.

Proof. The Jacobi field $Z(t)$, which is defined on $[t_0 - \delta, t_0 + \delta]$ has the property that $Z(t_0 - \delta) = X(t_0 - \delta)$ where X is a normal Jacobi field on $[0, t_0]$, and $Z(t_0 + \delta) = 0$. Thus $\langle Z(t_0 - \delta), \dot\gamma(t_0 - \delta) \rangle = \langle X(t_0 - \delta), \dot\gamma(t_0 - \delta) \rangle = 0$ as X was already shown to be normal. Likewise $\langle Z(t_0 + \delta), \dot\gamma(t_0 + \delta) \rangle = \langle 0, \dot\gamma(t_0 + \delta) \rangle = 0$.

By *Proposition 4.3* Z must be minimal. □

Exercise 6 Prove that the normal Jacobi field Z from the proof of *Theorem 4.1* from the note on Lectures 1-4 is minimal.

Before beginning the proof of the minimality of Z I will state and prove a lemma.

Lemma 0.1. *For any smooth normal vector field $W = f^i W_i$ written in terms of an orthonormal basis $\{W_1, \ldots, W_{n-1}\}$ for the space of normal Jacobi fields with a prescribed final value. If $W(1) = 0$ we can write $I(W)$ as*

$$I(W) = \int_0^1 \left| \dot f^i W_i \right|^2 \, dt - f^i(0) f^j(0) \left. \langle \nabla_{\dot\gamma} W_i, W_j \rangle \right|_{t=0}$$

Proof. The formula for I of a normal vector field is as shown in the notes

$$I(W) = \int_0^1 \left[\langle \nabla_{\dot{\gamma}} W, \nabla_{\dot{\gamma}} W\rangle - R(W, \dot{\gamma}, W, \dot{\gamma})\right] dt$$

Computing the first term

$$\begin{aligned}
\langle \nabla_{\dot{\gamma}} W, \nabla_{\dot{\gamma}} W\rangle &= \left\langle \dot{f}^i W_i + f^i \nabla_{\dot{\gamma}} W_i, \dot{f}^j W_j + f^j \nabla_{\dot{\gamma}} W_j \right\rangle \\
&= \left|\dot{f}^i W_i\right|^2 + 2\dot{f}^i f^j \langle W_i, \nabla_{\dot{\gamma}} W_j\rangle + f^i f^j \langle \nabla_{\dot{\gamma}} W_i, \nabla_{\dot{\gamma}}, W_j\rangle
\end{aligned}$$

Computing the second term

$$\begin{aligned}
R(W, \dot{\gamma}, W\dot{\gamma}) &= f^i f^j \langle R(W_i, \dot{\gamma})\dot{\gamma}, W_j\rangle \\
&= -f^i f^j \langle \nabla_{\dot{\gamma}} \nabla_{\dot{\gamma}} W_i, W_j\rangle
\end{aligned}$$

Where the last equality is because of the Jacobi equation applied to W_i. Noting

$$\frac{d}{dt} \langle \nabla_{\dot{\gamma}} W_i, W_j\rangle = \langle \nabla_{\dot{\gamma}} \nabla_{\dot{\gamma}} W_i, W_j\rangle + \langle \nabla_{\dot{\gamma}} W_i, \nabla_{\dot{\gamma}} W_j\rangle$$

We see that

$$\langle \nabla_{\dot{\gamma}} W, \nabla_{\dot{\gamma}} W\rangle - R(W, \dot{\gamma}, W, \dot{\gamma}) = \left|\dot{f}^i W_i\right|^2 + 2\dot{f}^i f^j \langle W_i, \nabla_{\dot{\gamma}} W_j\rangle + f^i f^j \frac{d}{dt} \langle \nabla_{\dot{\gamma}} W_i, W_j\rangle$$

Also we have

$$\frac{d}{dt} \left\langle f^i \nabla_{\dot{\gamma}} W_i, f^j W_j\right\rangle = \dot{f}^i f^j \langle \nabla_{\dot{\gamma}} W_i, W_j\rangle + f^j \dot{f}^i \langle \nabla_{\dot{\gamma}} W_j, W_i\rangle + f^i f^j \frac{d}{dt} \langle \nabla_{\dot{\gamma}} W_i, W_j\rangle$$

So we see that,

$$\langle \nabla_{\dot{\gamma}} W, \nabla_{\dot{\gamma}} W\rangle - R(W, \dot{\gamma}, W, \dot{\gamma}) = \left|\dot{f}^i W_i\right|^2 + \dot{f}^i f^j \left[\langle W_i, \nabla_{\dot{\gamma}} W_j\rangle + \langle \nabla_{\dot{\gamma}} W_i, W_j\rangle\right] + \frac{d}{dt} \left\langle f^i \nabla_{\dot{\gamma}} W_i, f^j W_j\right\rangle$$

Now to show that the middle term of the right hand side is zero

$$\begin{aligned}
\frac{d}{dt} \langle \nabla_{\dot{\gamma}} W_i, W_j\rangle &= \langle \nabla_{\dot{\gamma}} \nabla_{\dot{\gamma}} W_i, W_j\rangle + \langle \nabla_{\dot{\gamma}} W_i, \nabla_{\dot{\gamma}} W_j\rangle \\
&= \langle \nabla_{\dot{\gamma}} W_i, \nabla_{\dot{\gamma}} W_j\rangle - R(W_j, \dot{\gamma}, W_i, \dot{\gamma})
\end{aligned}$$

And likewise

$$\frac{d}{dt} \langle W_i, \nabla_{\dot{\gamma}} W_j\rangle = \langle \nabla_{\dot{\gamma}} W_i, \nabla_{\dot{\gamma}} W_j\rangle - R(W_j, \dot{\gamma}, W_i, \dot{\gamma})$$

And examining the middle term of the previous expansion

$$\frac{d}{dt} \left[\langle W_i, \nabla_{\dot{\gamma}} W_j\rangle - \langle \nabla_{\dot{\gamma}} W_i, W_j\rangle\right] = 0$$

So it is equal to a constant, but since $W_i(1) = W_j(1) = 0$ the whole term is equal to 0. And we can simplify

$$\langle \nabla_{\dot\gamma} W, \nabla_{\dot\gamma} W \rangle - R(W, \dot\gamma, W\dot\gamma) = \left| \dot f^i W_i \right|^2 + \frac{d}{dt} \left\langle f^i \nabla_{\dot\gamma} W_i, f^j W_j \right\rangle$$

And integrating from 0 to 1 gives the desired result

□

Now we return to the proof of the minimality of Z.

Proof. Z is in particular a smooth normal field, so it can be written in terms of an orthonormal basis $\{W_1, \ldots, W_{n-1}\}$ for the space of normal Jacobi fields with perscribed initial value $Z(t_0 - \delta) = X(t_0 - \delta)$. Let $Z(t) = f^i W_i$. Since Z is a Jacobi field, it is a solution of the Jacobi equation. Evaluating the Jacobi equation for $f^i W_i$ and using the fact that each W_i satisfies the Jacobi equation gives

$$\begin{aligned} 0 = \nabla_{\dot\gamma} \nabla_{\dot\gamma} (f^i W_i) + R(\dot\gamma, f^i W_i)\dot\gamma &= \nabla_{\dot\gamma} \left(\dot f^i W_i + f^i \nabla_{\dot\gamma} W_i \right) + f^i R(\dot\gamma, W_i)\dot\gamma \\ &= \ddot f^i W_i + 2\dot f^i \nabla_{\dot\gamma} W_i + f^i \nabla_{\dot\gamma} \nabla_{\dot\gamma} W_i + f^i R(\dot\gamma, W_i)\dot\gamma \\ &= \ddot f^i W_i + 2\dot f^i \nabla_{\dot\gamma} W_i \end{aligned}$$

This implies that $\dot f^i = 0 \quad \forall i$. If not, then $\nabla_{\dot\gamma} W_i$ would be some multiple of W_i which would contradict the fact that $Z(t_0 + \delta) = 0$. Thus f^i is a constant for each i.
By the lemma, $I(W)$ can be written

$$I(W) = \int_0^1 \left| \dot f^i W_i \right|^2 \, dt - f^i(0) f^j(0) \left. \langle \nabla_{\dot\gamma} W_i, W_j \rangle \right|_{t=0}$$

And for $Z = f^i W_i$ where the coefficients are constant the formula reduces to

$$I(Z) = -f^i(t_0 - \delta) f^j(t_0 - \delta) \left. \langle \nabla_{\dot\gamma} W_i, W_j \rangle \right|_{t = t_0 - \delta}$$

And for any smooth field $W = g^i W_i$ which agrees with Z on $t_0 - \delta$ and $t_0 + \delta$ we have

$$I(W) - I(Z) = \int_{t_0 - \delta}^{t_0 + \delta} \left| \dot g^i W_i \right|^2 \, dt \geq 0$$

So Z is strictly minimal amongst smooth fields with the same endpoints. □