Lecture notes on Green function on a Remannian Manifold

Nov. 26th

Let $(\bar{M}, g)$ be a compact Riemannian manifold possibly with boundary. $M \subset \bar{M}$ with $M, \partial M$ oriented. We define the following differetial operator on the Riemannian manifold:

$$L = \Delta_g + aI$$

Here $a \in L^\infty(M)$ and $\Delta_g u = -div_g(\nabla u)$ (Or for simplicity you can choose $a \in C^\infty$). And we define the kernel K_a (depending on the constant a) of the differential operator L:

$$K_a = KerL \cap W_0^{1,2}(M)$$

It is obvious that if the differential operator is coersive, then the Kernel $K_a = 0$.

Definition 1. *(Green Function) Define $G : \bar{M} \times \bar{M} \backslash diag\{\bar{M}\} \to \mathbb{R}$ is a Green function for L with Dirichelet Boundary condition if $\forall x \in M$:*

(i) $G(x, \cdot) = G_x \in L^1(M)$

(ii) $G_x \perp K_a$

(iii) if $\varphi \in C_c^2(\bar{M})$, then:

$$\int_M G_x L\varphi dv_g = (\varphi - \pi_a(\varphi))(x)$$

Here $\pi_a : W_0^{1,2}(M) \to K_a$ is the orthogonal projection to the Kernel of the differential operator L.

For the Green function, we have the following Theorem:

Theorem 1. *Suppose $a \in L^\infty$(or $C^\infty$for simplicity). There exists a unique green function with respect to the differential operator L as in the above definition. Moreover, we have the following property:*

(i) $\int G_x L\varphi dv_g = (\varphi - \pi_a(\varphi))(x) + \int_{\partial M} \frac{\partial G_x}{\partial \nu} \varphi d\sigma_g$

(ii) $G_x(y) = 0$, if $x \in \partial M$. And for all $x \in M$, the following equation is satisfied:

$$LG_x = 0, y \in M\backslash\{x\}$$
$$G_x = 0, y \in \partial M$$

(iii) Because we are basically concerning about compact manifolds without boundaries, to simplify our life a little bit, we set $\partial M = \phi$ (empty). And by convention, set $d_g(x, \partial M) = dist_g(x, \partial M) = 1$, we will have the following bound:

$$-C_3 d_g(x, \partial M) d_g(y, \partial M) + \frac{C_1}{d_g(x,y)^{n-2}} \leq G(x,y) \leq \frac{C_2}{d_g(x,y)^{n-2}}$$

notice that if the differential operator is coersive here, then the C_3 will be zero

(iv) $|\nabla G|$ also satisfy a similar estimate:

$$|\nabla G(x,y)| \leq C_1 d_g(x,y)^{1-n}$$

Proposition 1. *There exists $f_n, H, l \in C^\infty(\bar{M}\times\bar{M}\backslash diag\bar{M})$ such that the following equation holds (Notation: $H(x,\cdot) = H_x$):*

(*)

$$\int_M H_x \triangle_g \varphi dv_g = \varphi(x) + \int_M l_x\varphi + \int_{\partial M} -\partial_\nu \varphi H_x + \varphi \partial_\nu H_x$$

Here we denote $I = \varphi(x)$, $II = \int_M l_x\varphi$, and $III = \int_M -\partial_n u\varphi H_x + \varphi\partial_\nu H_x$. And you can think of $l_x = \Delta_g H_x$, H_x our Green function.

and:

i) $x, y \in M$, $x \neq y$,

$$d_g(x,y)^{n-2}|H(x,y)| + d_g(x,y)^{n-1}|\nabla H_x| \leq C_M$$

ii) $d_g(x,y)^{n-2}|l(x,y)| \leq C_M$ for some $C_M > 0$.

Proof. Here let η be the cut off function $\eta(x,y) = \eta_0(\frac{d_g(x,y)}{\delta})$, η_0 here is the standard bumping function, let $\delta_x = \frac{\text{injective radius}}{2}$, $\delta = inf\{\delta_x\}$, $H(x,y) = \frac{\eta(x,y)}{(n-2)nVol(B_1)d_g(x,y)^{n-2}}$, ($Vol(B_1)$ here is just the volume of the Euclidean ball) which is just the usual fundamental solution with a cut off function η. Now we use the same calculation as last time, we will have the following:

$$\int_M H_x \Delta\varphi = I + III + \int_M \varphi\Delta_g H_x$$

Exercise: Show (ii) in the theorem. □

Proposition 2. *Suppose $a, h \in L^\infty(M)$. And $\Gamma, f \in L^\infty_{loc}(\overline{M}\times\overline{M}\backslash Diag(\overline{M}))$ are given functions, such that the following conditions are satisfied:*

i) $\Gamma_x \in C^1(\overline{M}\backslash\{x\})$, $|\Gamma(x,y)| \leq C_1 d_g(x,y)^{2-n}$ and $|\nabla\Gamma_x(y)| \leq C_1 d_g(x,y)^{1-n}$ for all $x, y \in \overline{M}, x \neq y$

ii) $|f| \leq C_1 d_g(x,y)^{2-n}$ for all $x, y \in \overline{M}$, $x \neq y$.

iii)And suppose for all $\varphi \in C^2(\overline{M})$ and for all $x \in M$, the following equation is satisfied:

$$\int_M (\Delta_g\varphi + h\varphi)\Gamma_x dv_g = \varphi(x) + \int_M f_x\varphi dv_g + \int_{\partial M}(-\partial_\nu\varphi\Gamma_x + \varphi\partial_\nu\Gamma_x)d\sigma_g \tag{1}$$

Then there exists $\widehat{G_x} \in C^\infty(M\times M\backslash Diag(M))\bigcap L^1(M)$ such that for all $x \in M$, $\widehat{G_x} \perp K_a$ and the following equation holds:

$$\int_M (\Delta_g\varphi + a\varphi)\widehat{G_x}dv_g = (\varphi - \pi_a(\varphi))(x) + \int_{\partial M}(-\partial_\nu(\varphi - \pi_a(\varphi))\widehat{G_x} + \varphi\partial_\nu\widehat{G_x})d\sigma_g \tag{2}$$

Proof.

Remark 1. *In the first step of the proof, we use the following idea: First, we already have a good candidate, which is the Euclidean Green function with cut-off, i.e., $G_0 = H$. When we apply the Laplacian on this object, an extra residue term will come up. That is:*

$$\Delta G_0 = \delta + R_1$$

Here δ is the Dirac mass and the R_1 is the residue. What we want to do now is to correct the original Green function. In order to do that, we introduce a correction function G_1 satisfying:

$$\Delta G_1 = R_1$$

Formally speaking, $G_1 = G * R_1$, here G is the actual Green function on the manifold. If we know how to solve for G_1, then $G_0 - G_1$ will be our Green function. But now the problem is that we don't know G, so we cannot solve for G_1 directly. So we take $G_1 = G_0 * R_1$. Now we have that:

$$\Delta(G_0 * R_1) = (\delta + R_1) - \Delta G_0 * R_1 = (\delta + R_1) - (\delta + R_1)R_1 = \delta - R_1^2$$

Now we can do the same thing as before, approximate the solution of the following equation:

$$\Delta G_2 = -R_1^2$$

Take $G_2 = G_0 * (-R_1^2)$*, then* $\Delta(G_0 - G_1 + G_2) = \delta + R_1^3 ...$ *And we want to prove that the* R_1^n *is small at some stage.*

Step 1: Now suppose there exist $\Gamma_1, ...\Gamma_k : \overline{M} \times \overline{M} \backslash Diag(\overline{M}) \to \mathbb{R}$ such that there is $C_2 > 0$ such that for all $i \in \{1, ..., k\}$, on a

$$\Gamma_i \in L^{\infty}_{loc}(M \times M \backslash Diag(M)) \text{ and } |\Gamma_i(x, y)| \leq C_2 d_g(x, y)^{2-n} \tag{3}$$

for all $x, y \in \overline{M}, x \neq y$. (In the smooth setting, you can choose them to be smooth) Now define the following quantity (' here is not derivative):

$$G'(x) := \Gamma(x, y) + \sum_{i=1}^{k} \int_M \Gamma_i(x, z)\Gamma(z, y) dv_g(z)$$

for all $x, y \in \overline{M}, x \neq y$. Now due to the de Giraud Lemma (Hardy-Little-Sobolev Lemma (See Appendix)), we have $G' \in L^{\infty}_{loc}(M \times M \backslash Diag(M))$ and that $|G'(x, y)| \leq C(M, C_1, C_2) d_g(x, y)^{2-n}$ for all $x, y \in \overline{M}$ with $x \neq y$. In particular, on a $G'_x \in L^1(M)$ for all $x \in M$. What's more, from the Lebesgue Dominated convergence theorem we have $G'_x \in C^{1,\theta}(\overline{M} \backslash \{x\})$ for all $x \in M$ for all $\theta \in (0, 1)$.

Calculate $\Delta_g G'_x + aG'_x$ in the sense of distributions. Set $\varphi \in C^2(\overline{M})$. We have:

$$\begin{aligned}
&\int_M (\Delta_g \varphi + a\varphi) G'_x dv_g = \int G'_x(\Delta_g \varphi + h\varphi) dv_g + \int_M G'_x(a - h) dv_g \\
=& \int_M \Gamma_x(\Delta_g \varphi + h\varphi) dv_g \\
+& \sum_{i=1}^{k} \int_{M \times M} \Gamma_i(x, z)\Gamma(z, y)(\Delta_g \varphi + h\varphi)(y) dv_g(y) dv_g(z) \\
+& \int_M G'_x(a - h) dv_g
\end{aligned}$$

here we apply the Fubini-Tornelli Theorem. Again using Fubini theorem on equation (1), one obtained:

$$\begin{aligned}
&\int_M (\Delta_g \varphi + a\varphi) G'_x dv_g \\
=&\varphi(x) + \int_M f_x \varphi dv_g + \int_{\partial M} (-\partial_\nu \varphi \Gamma_x + \varphi \partial_\nu \Gamma_x) d\sigma_g \\
&+ \sum_{i=1}^{k} \int_M \Gamma_i(x, z)(\varphi(z) + \int_M f_z \varphi dv_g + \int_{\partial M} (-\partial_\nu \varphi \Gamma_z + \varphi \partial_\nu \Gamma_z) d\sigma_g) \\
&+ \int_M G'_x(a - h) dv_g
\end{aligned}$$

Now condition (i) on the gradient of the function Γ will justify the following usage of the Fubini theorem:

$$\int_M (\Delta_g \varphi + a\varphi) G'_x dv_g = \varphi(x) + \int_M f_x \varphi dv_g + \sum_{i=1}^{k} \int_M \Gamma_i(x,\cdot)\varphi dv_g$$
$$+ \sum_{i=1}^{k} \int_M (\int_M \Gamma_i(x,z) f(z,y) dv_g(z))\varphi(y) dv_g(y)$$
$$\int_{\partial M} (-\partial_\nu \varphi G'_x + \varphi \partial_\nu G'_x) d\sigma + \int_M G'_x (a-h) dv_g$$

for the last term, it suffices to use the above definition for G'_x and get:

$$\begin{aligned}
&\int_M (\Delta_g \varphi + a\varphi) G'_x dv_g \\
=&\varphi(x) + \int_M (f_x + (a-h)\Gamma_x)\varphi dv_g + \sum_{i=1}^{k} \int_M \Gamma_i(x,\cdot)\varphi dv_g \\
&\sum_{i=1}^{k} \int_M (\int_M \Gamma_i(x,z)(f(z,y) + (a-h)(y)\Gamma(z,y)) dv_g(z))\varphi(y) dv_g(y) \\
&+ \int_{\partial M} (-\partial_\nu \varphi G'_x + \varphi \partial_\nu G'_x) d\sigma
\end{aligned} \tag{4}$$

By definition,

$$\Gamma_1(x,y) := -[f(x,y) + (a-h)(y)\Gamma(x,y)] \tag{5}$$
$$\Gamma_{i+1}(x,y) := \int_M \Gamma_i(x,z)\Gamma_1(z,y) dv_g(z) \tag{6}$$

for all $i \geq 1$ and for all $x, y \in M, x \neq y$. There exists $K, K' \geq 0$ such that

$$|a(x)| \leq K$$
$$|h(x)| \leq K'$$

for all $x \in M$.

Then by the standard de Giraud Lemma (The Hardy-Littlewood-Sobolev lemma), there exists $C_i(M, C_1, K, K')$ s.t.

$$|\Gamma_i(x,y)| \leq C_i(M, C_1, K, K') \begin{cases} d_g(x,y)^{2i-n} & \text{if } i < \frac{n}{2} \\ 1 + |\ln d_g(x,y)| & \text{if } i = \frac{n}{2} \\ 1 & \text{if } i > \frac{n}{2} \end{cases} \tag{7}$$

for all $x, y \in M, x \neq y$. In particular, Γ_i satisfy (3) for all $i \geq 1$. Then apply (4,5,6), we get the following equation:

$$\int_M (\Delta_g \varphi + a\varphi) G'_x dv_g = \varphi(x) + \int_{\partial M} (-\partial_\nu \varphi G'_x + \varphi \partial_\nu G'_x) d\sigma - \int_M \Gamma_{k+1}(x,\cdot)\varphi dv_g \tag{8}$$

Now choose $k = E(\frac{n}{2})$, so that $k + 1 > \frac{n}{2}$: now we define $\gamma := \Gamma_{k+1}$. Otherwise, due to the de Giraud Lemma(H.L.S lemma), there exists $C(M, K) > 0$ such that

$$|\int_M \Gamma_i(x,z)\Gamma(z,y) dv_g(z)| \leq C(M, C_1, K, K') d_g(x,y)^{3-n}$$

for all $x, y \in M, x \neq y$ and all $i \geq k+1$. So we obtain

$$|G'(x,y) - \Gamma(x,y)| \leq C(M, C_1, K, K')d_g(x,y)^{3-n} \tag{9}$$

for all $x, y \in M, x \neq y$. In particular,

$$|G'(x,y)| \leq C(M, C_1, K, K')d_g(x,y)^{2-n} \tag{10}$$

for all $x, y \in M, x \neq y$

Step 2. suppose $u' \in H^2_{1,0} \bigcap K_a^{\perp}$ is the unique weak solution of the following:

$$\begin{cases} \Delta_g u'_x + a u'_x = \gamma_x - \pi_a(\gamma_x) & \text{if } x \in M \\ u'_x = 0 & \text{if } x \in \partial M \end{cases}$$

Now it follows from standard elliptic theory that u'_x is unique and well defined. What's more, also due to the regularity theory, we have $u'_x \in H^p_2(M) \bigcap C^{1,\theta}(\overline{M})$ for all $p \geq 1$ and $\theta \in (0,1)$ and there will exist $C > 0$ such that (of course, if you are in the smooth setting, everything here will be smooth):

$$||u'_x||_{C^1} \leq C(M, K)(||\gamma_x - \pi_a(\gamma_x)||_\infty + ||u'_x||_2) \tag{11}$$

as $u'_x \in K_a^{\perp}$, or $\pi_a(u'_x) = 0$ it can be shown that(Here we have applied the estimate $\int_M (\Delta_g \varphi + a\varphi)^2 dv_g \geq \lambda ||\varphi - \pi_a(\varphi)||_2^2$, which we will not be able to prove here):

$$\lambda ||u'_x||_2^2 \leq ||\gamma_x - \pi_a(\gamma_x)||_2^2 \tag{12}$$

From (11, 12) we have

$$||u'_x||_{C^1} \leq C(M, K, \lambda)||\gamma_x - \pi_a(\gamma_x)||_\infty \tag{13}$$

Equations (7,13) together with the estimate $\pi_a(f) \leq C(M, K, d)||f||_1$ (estimate (13) in Frederic's notes) will give us the following:

$$||u'_x||_{C^1} \leq C(M, K, \lambda, d)||\gamma_x||_\infty \leq C'(M, K, K', C_1, \lambda, d) \tag{14}$$

We can define:

$$u_x := u'_x - \sum_{i=1}^{d_a} \left(\int_M (G'_x + u'_x)\psi_i dv_g\right)\psi_i \tag{15}$$

here the ψ_i's are defined as follows: Because K_a is finite dimensional $d_a \leq d$, we can find the orthonormal bases $\{\psi_1, \psi_2, ...\psi_{d_a}\}$ for this subspace, the ψ_i's satisfy the following equation:

$$\begin{cases} \Delta_g \psi_i + a\psi_i = 0 & \text{on } M \\ \psi_i = 0 & \text{on } \partial M \end{cases}$$

Therefore, $u_x \in C^1(\overline{M})$ satisfies:

$$\begin{cases} \Delta_g u_x + a u_x = \gamma_x - \pi_a(\gamma_x) & \text{on } M \\ u_x = 0 & \text{on } \partial M \end{cases}$$

Now from (1014) and the estimate for the ψ_i: $||\psi_i||_{C^{1,\theta}} \leq C(M,K,\theta)||\psi_i||_2 = C(M,K,\theta)$ ((14)on Frederic's notes), we have

$$
\begin{aligned}
&|| \sum_{i=1}^{d_a} (\int_M (G'_x + u'_x)\psi_i dv_g)\psi_i||_{C^1} \leq \sum_{i=1}^{d_a} |\int_M (G'_x + u'_x)\psi_i dv_g| \cdot ||\psi_i||_{C^1} \\
&\leq C(M,K) \sum_{i=1}^{d_a} (||G'_x||_1 + ||u'_x||_1)||\psi_i||_\infty \leq C(M,K,K',C_1,\lambda,d)
\end{aligned}
$$

Step 3. Now define

$$
\hat{G}(x,y) := G'(x,y) + u_x(y) \text{for} x \in M, y \in \overline{M}, x \neq y \tag{16}
$$

In particular, for all $x \in M$, and a $G_x \in L^1(M)\bigcap C^1(\overline{M}\backslash\{x\})$. And $\varphi \in C^2(\overline{M})$. As $u_x \in H_2^p(M)$ for all $p \geq 1$, by integration by part, it follows from (8,16) that

$$
\begin{aligned}
&\int_M (\Delta_g \varphi + a\varphi)\widehat{G_x} dv_g \\
=&\varphi(x) + \int_{\partial M} (-\partial_\nu \varphi G'_x + \varphi \partial_\nu \hat{G}'_x) d\sigma - \int_M \gamma_x \varphi dv_g + \int_M u_x(\Delta_g \varphi + a\varphi) dv_g \\
=&\varphi(x) + \int_{\partial M} (-\partial_\nu \varphi (G'_x + u_x) + \varphi \partial_\nu (G'_x + u_x)) d\sigma + \int_M \varphi(\Delta_g u_x + a u_x - \gamma_x) dv_g \\
=&\varphi(x) + \int_{\partial M} (-\partial_\nu \varphi (G'_x + u_x)) d\sigma - \int_M \varphi \pi_a(\gamma_x) dv_g \\
=&\varphi(x) + \int_{\partial M} (-\partial_\nu \varphi \hat{G}_x) d\sigma - \int_M \varphi \pi_a(\gamma_x) dv_g
\end{aligned}
$$

As $\pi_a(\gamma_x) \in K_a = span\{\psi_1, \psi_2, ...\psi_{d_a}\}$, there exists $c_i(x), i \in \{1, ...d_a\}$ such that $\pi_a(\gamma_x) = \sum_{i=1}^{d_a} c_i(x)\psi_i$. Thus one have the formula:

$$
\int_M (\Delta_g \varphi + a\varphi)\hat{G}_x dv_g = \varphi(x) + \int_{\partial M} (-\partial_\nu \varphi \hat{G}_x + \varphi \partial_\nu \hat{G}_x) d\sigma - \sum_{i=1}^{d_a} c_i(x) \int_M \varphi \psi_i dv_g \tag{17}
$$

Fix $i \in \{1, ...d_a\}$. Now apply the equation (17) for ψ_i, then utilise the fact that $\{\psi_1, ...\psi_{d_a}\}$ is the orthonormal bases of the K_a and $\psi_i = 0$ on ∂M, one obtain:

$$
\begin{aligned}
0 =& \int_M (\Delta_g \psi + a\psi)\widehat{G_x} dv_g \\
=&\psi_i(x) - \int_{\partial M} \partial_\nu \psi_i \hat{G}_x d\sigma - \sum_{i=1}^{d_a} c_j(x) \int_M \psi_i \psi_j dv_g \\
=&\psi_i(x) - c_i(x) - \int_{\partial M} \partial_\nu \psi_i \hat{G}_x d\sigma
\end{aligned} \tag{18}
$$

And therefore $c_i(x) = \psi_i(x) - \int_{\partial M} \partial_\nu \psi_i \hat{G}_x d\sigma$ for all $i \in \{1, ...d_a\}$ (Notice that ψ_i is not C^2, here we utilise $\psi_i \in H_2^p(M)$ for all $p \geq 1$). Now return to the formula (17), we obtain

$$\begin{aligned}
&\int_M (\Delta_g \varphi + a\varphi)\widehat{G_x} dv_g \\
=&\varphi(x) + \int_{\partial M} (-\partial_\nu \varphi \hat{G}_x + \varphi \partial_\nu \hat{G}_x) d\sigma - \sum_{i=1}^{d_a} (\int_M \varphi \psi_j dv_g)\psi_i(x) \\
&+ \int_{\partial M} \partial_\nu (\sum_{i=1}^{d_a} (\int_M \varphi \psi_j dv_g)\psi_i(x))\hat{G}_x d\sigma \\
=&(\varphi - \pi_a(\varphi))(x) + \int_{\partial M} (-\partial_\nu (\varphi - \pi_a(\varphi))\hat{G}_x + \varphi \partial_\nu \hat{G}_x) d\sigma
\end{aligned} \tag{19}$$

for all $x \in M$. Hence we have proved (1)

Step 4. Now we study the orthogonality relations. For $i \in \{1, ...d_a\}$. We have

$$\begin{aligned}
&\int_M \widehat{G_x} \psi_i dv_g \\
=&\int_M (G'_x + u'_x - \sum_{i=1}^{d_a} (\int_M (G'_x + u'_x)\psi_j dv_g)\psi_j)\psi_i dv_g \\
=&\int_M (G'_x + u'_x)\psi_i dv_g - \sum_{i=1}^{d_a} (\int_M (G'_x + u'_x)\psi_j dv_g) \int_M \psi_i \psi_j dv_g \\
=&\int_M (G'_x + u'_x)\psi_i dv_g - \int_M (G'_x + u'_x)\psi_i dv_g = 0
\end{aligned} \tag{20}$$

Because the family $\{\psi_1, ...\psi_{d_a}\}$ spans K_a, one now derive:

$$\hat{G}_x \perp K_a \tag{21}$$

Step 5. Now we show the estimated point. Due to (9) and (18), there exists $C(M, K, K', \lambda, d) > 0$ such that

$$|\hat{G}_x(x, y) - \Gamma(x, y)| \le C(M, K, K', C_1, \lambda, d) d_g(x, y)^{3-n}$$

for all $x, y \in M, x \neq y$. This proves (2) and finishes the proof of the proposition. □

Appendix.

Theorem 2. *(Hardy-Littlewood-Sobolev fractional integration inequality) Let $d \ge 1, 0 < s < d$, and $1 < p < q < \infty$ be such that*

$$\frac{d}{q} = \frac{d}{p} - s$$

Then we have

$$|| \int_{\mathbb{R}^d} \frac{f(x)}{|x - y|^{d-s}} ||_{L^q_y(\mathbb{R}^d)} \lesssim_{d,s,p,q} ||f||_{L^p_x(\mathbb{R}^d)}$$

for all $f \in L^p_x(\mathbb{R}^d)$

The proof of the theorem requires the following two theorems (one of the is the basic content of harmonic analysis). This idea of the proof is sketched in the exercises of Terrence Tao's notes on Fourier Analysis.

Theorem 3. *(Hardy-Littlewood maximal inequality) Define the Hardy-Littlewood maximal operator as follows:*

$$Mf := sup_{r>0}\frac{1}{|B(x,r)|}\int_{B(x,r)} f(y)dy$$

We have that:

$$||Mf||_{L^p(\mathbb{R}^d)} \lesssim_{p,d} ||f||_{L^p}$$

for any $1 < p \leq \infty$ *and any* $f \in L^p(\mathbb{R}^d)$*, and also*

$$||M||_{L^{1,\infty}(\mathbb{R}^d)} \lesssim_d ||f||_{L^1(\mathbb{R}^d)}$$

for any $f \in L^1(\mathbb{R}^d)$

Remark 2. *Here the* $L^{1,\infty}$ *norm is the Lorentz space norm. Here because we don't need that part of the theorem, we will not give the exact definition for that here.*

Proof. You can find the proof on any Harmonic Analysis book. □

Theorem 4. *(Hedberg's inequality) Let* $1 \leq p < \infty$, $0 < \alpha < d/p$*, and let* f *be locally integrable on* $\mathbb{R}^d$*. The following inequality hold:*

$$\int_{\mathbb{R}^d} \frac{|f(y)|}{|x-y|^{d-\alpha}}dy \lesssim_{d,\alpha,p} ||f||_{L^p(\mathbb{R}^d)}^{\alpha p/d}(Mf(x))^{1-\frac{\alpha p}{d}}$$

Proof. Notice that there are three symmetries available for this estimate: translation $f(x) \mapsto f(x-x_0)$, homogeneity $f(x) \mapsto c(f(x))$ and scaling $f(x) \mapsto f(x/\lambda)$. Using all the three we can normalize $x, ||f||_{L^p(\mathbb{R}^d)}, Mf(x)$ to be 0, 1 and 1 respectively. So we just have to show the following inequality hold:

(*)

$$\int_{\mathbb{R}^d} \frac{|f(y)|}{|y|^{d-\alpha}}dy \lesssim 1$$

Now separate the $\mathbb{R}^d$ into annulus $A_n = \{y|2^n < |y| \leq 2^{n+1}\}, \mathbb{R}^n = \bigcup_{-\infty}^{\infty} A_n$, we can now rewrite the (*) by

$$\sum_{-\infty}^{\infty}\int_{A_n}\frac{|f(y)|}{|y|^{d-\alpha}}dy$$

Now we estimate the $I = \sum_0^\infty \int_{A_n} \frac{|f(y)|}{|y|^{d-\alpha}}dy$ and $II = \sum_0^\infty \int_{A_n} \frac{|f(y)|}{|y|^{d-\alpha}}dy$ separatively.

For I, we can just make use of the Hölder inequality, which tells you the following:

$$\int_{A_n}\frac{|f(y)|}{|y|^{d-\alpha}}dy \leq (\int_{A_n}|f(y)|^p dy)^{1/p}(\int_{A_n}\frac{1}{|y|^{d-\alpha}}dy)^{1/q} = ||f||_{L^p(A_n)}(\omega_d r^{d-dq+\alpha q}|_{2^n}^{2^{n+1}})^{1/q} \leq ||f||_{L^p(A_n)}$$

So we will have the following:

$$I \lesssim 1$$

Because we have already normalized the L^p norm of f.

Now we estimate the II part ($n \leq 0$):

$$\int_{A_n} \frac{|f(y)|}{|y|^{d-\alpha}} dy = (\int_{A_n} \frac{|f(y)|}{2^{nd}} dy)(\int_{A_n} \frac{|2^{nd}|}{|y|^{d-\alpha}} dy) = \omega_d(\int_{A_n} \frac{|f(y)|}{\omega_d 2^{nd}} dy)(\frac{2^{nd}}{2^{n(d-\alpha)}}) \lesssim Mf(0)2^{\alpha n}$$

Note that $n < 0$, after summing all the terms, we have the following:

$$II \lesssim Mf(0) \lesssim 1$$

In a word, we now have $I + II \lesssim 1$. After doing the back normalization, we finish the proof of the Hedberg's inequality. □

With these two estimates, we can easily derive the Hardy-Littlewood-Sobolev inequality.