MATH 742 HEAT EQUATION AND KERNEL

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1. Second-Order Parabolic Equations

Second-order parabolic equations are natural generalizations of the heat equation and we will study in this section the existence, uniqueness, and regularity of appropriately defined weak solutions.

1.1. **Formulation of Weak Solutions.**

1.1.1. *Notations.* In this note, we assume Ω to be an open, bounded domain in $\mathbb{R}^n$, and set $\Omega_T = \Omega \times (0, T]$.

We study the following initial/boundary-value problem

$$\begin{cases} u_t + Lu = f, & \text{in } \Omega_T \\ u = 0, & \text{on } \partial\Omega \times [0, T] \\ u = g, & \text{on } \Omega \times \{t = 0\} \end{cases} \tag{1.1}$$

where $f(x, t) : \Omega_T \to \mathbb{R}$ and $g(x) : \Omega \to \mathbb{R}$ are given with $u(x, t) : \overline{\Omega}_T \to \mathbb{R}$ the unknown function. L here is a time-independent second order differential operator in divergence form

$$Lu = -\partial_j(a^{ij}\partial_i u) + b^i\partial_i u + cu \tag{1.2}$$

for given coefficients a^{ij}, b^i, c. Note that we assume the summation convention for upper and lower indices.

We require that the differential operator L to be uniformly elliptic, i.e. there exists a constant $\theta > 0$ such that

$$a^{ij}(x)\xi_i\xi_j \geq \theta|\xi|^2 \tag{1.3}$$

for all $x \in \Omega, \xi \in \mathbb{R}^n$. Also, we assume self-adjointness of L by requiring $a^{ij} = a^{ji}$.

1.1.2. *Weak Solutions.* In order to find appropritate notion of weak solution to initial/boundary-value problem (1.1), we first assume that

$$a^{ij}, b^i, c \in L^\infty(\Omega), \quad f \in L^2(\Omega_T), \quad g \in L^2(\Omega)$$

Also for $u, v \in H_0^1(\Omega)$, we have the following time-independent bilinear form

$$B[u, v] := \int_\Omega a^{ij}\partial_i u\partial_j v + b^i\partial_i uv + cuvdx \tag{1.4}$$

Further more, to better accomodate this evolution problem, we consider $u(x, t)$,$f(x, t)$, $u'(x, t)$ as mappings from $[0, T]$ into the functional triplet $H_0^1(\Omega) \subset L^2(\Omega) \subset H^{-1}(\Omega)$. Now we can state the following

Definition 1.1. A fucntion

$$u \in L^2(0,T;H_0^1(\Omega)) \text{ with } u' \in L^2(0,T;H^{-1}(\Omega))$$

is a weak solution of the parabolic initial-boundary problem (1.1) provided

$$< u', v >_{H^{-1}} + B[u,v;t] = (f,v)_{L^2} \tag{1.5}$$

for each $v \in H_0^1(\Omega)$ and a.e. $0 \le t \le T$, and

$$u(x,0) = g. \tag{1.6}$$

Remark 1.2. For more details about the functional spaces $L^2(0,T;H_0^1(\Omega))$ and $L^2(0,T;H^{-1}(\Omega))$, we refer the reader to Evans 5.9.2.

Remark 1.3. From Thm 3 in Evans 5.9.2, we know that $u \in C([0,T];L^2(\Omega))$, hence equation (1.6) makes sense.

1.2. **Existence and Uniqueness.** Now we state the general existence and uniqueness result for the initial/boundary problem (1.6) with $f = 0$.

Theorem 1.4. *Given $g \in L^2(\Omega)$ and L as described above, there is a unique u that solves the initial/boundary problem (1.1) in the sense of a weak solution. Furthermore, if $a^{ij} \in C^\infty(\Omega)$, then $u \in C^\infty(\Omega \times (0,T])$.*

Proof. Key fact: the exponential decay of eigenvalues! First, let's formally look for separated-variable solutions $a(t)b(x)$. By computation, we see that $e^{-\lambda_j t}\varphi_j$ for any fixed $j \ge 1$ is such a solution. Here λ_j is the eigenvalue of L as an elliptic operator and φ_j the corresponding complete orthonormal set of eigenfunctions. To solve our initial/boundary value problem (1.1), we use the superposition principle, hence we look for

$$u(x,t) = \sum_{j=1}^{\infty} c_j e^{-\lambda_j t}\varphi_j(x) \tag{1.7}$$

Due to the initial condition it is clear that $c_j = (g(x), \varphi_j)_{L^2(\Omega)}$.

Given the following estimate that $\lambda_j \ge C^{-1} j^{\frac{2}{n}}$ for sufficiently large j and the fact that $\sum_{j=1}^{\infty} c_j^2 < \infty$, we can see that for any fixed $t > 0$, $u(x,t)$ is a well defined function in $H_0^1(\Omega)$, and because of the uniform boundedness of its H^1-norm, the series converges to a function in $L^2((0,T], H_0^1(\Omega))$. Similarly we know that its weak time derivative $u'(x,t)$ also belongs to this space. What remained to prove is that $\lim_{t\to 0} u(x,t) \to g(x)$ in L^2-norm and that $u and u'$ satisfy equation (1.5), which can be easily checked. Hence finishes the proof of existence. □

The proof for uniqueness is left in class an exercise.

1.3. **Regularity.** Now we study the smoothness of our solution to problem (1.1).

Note that due to the negative exponentials in the power series

$$u(x,t) = \sum_{j=1}^{\infty} c_j e^{-\lambda_j t}\varphi_j(x) \tag{1.8}$$

defining our solution, if we have regularity of each of the eigenfucntions φ_j in an uniform fashion, then we can differentiate this series termwise as many times as we want as long as $t > 0$, hence the power series solution belongs to $C^\infty(\Omega \times (0,T))$. For this, we have the following regularity of φ_j based the elliptic regularity theory.

Lemma 1.5. *If φ_j are a complete orthonormal set of eigenfunctions for the operator L, where we assume the coercivity condition (C) and that the coefficients a^{ij} are bounded and of C^∞-class, then for any ball $B_\rho(y) \subset \Omega$ and any $\theta \in (0,1)$*

$$||\varphi_j||_{q,B_{\theta\rho}(y)} \le C(\theta,\rho,a^{ij}) j^{\frac{2q}{n}}, \quad q \ge 1, j \ge 1 \tag{1.9}$$

Furthermore, if we have regularity of $\partial\Omega$ and $a^{ij} \in C^\infty(\overline{\Omega}^\infty)$, our φ_j will have similar regularity up to the boundary of Ω.

Remark 1.6. For any integer $l \ge 0$, we can find a q such that $n/2+l \le q \le n/2+l+1$, then by Sobolev embedding theorem, we know that φ_j is of class C^l, hence of C^∞-class.

Now we the smoothness of the eigenfunctions as required, hence that of the solution.

1.4. **Heat Kernel and Weyl's Theorem.** Note that the above arguments can all be modified to the case of Borel measure μ of compact support in place of the initial data g, and in the particular case where $\mu = \delta_y$ for any fixed $y \in \Omega$, then the solution u is given by

$$p(x,y,t) := \sum_{j=1}^{\infty} e^{-\lambda_j t}\varphi_j(y)\varphi_j(x) \tag{1.10}$$

for $x \in \Omega$ and $t > 0$.

This fucntion $p(x,y,t)$ is called the heat kernel for the operator L on the domain Ω.

Remark 1.7. The heat kernel is unique as a consequence that the solution to (1.1) is unique.

In case where a^{ij} is smooth on the domain Ω, we can approximate the heat kernel by a sequence of functions $\{q_i\}$ in $C^\infty(\Omega \times [0,\infty))$: take any sequence $\psi_i \in C_c^\infty(\Omega) \to \delta_y$ in the sense of Borel measure with y fixed, thus $(\psi_i,\psi)_{L^2} \to \psi(y)$ for each $\psi \in C_c^\infty(\Omega)$. We also require that the support of ψ_i to be within the radius $1/i$ disk around y with $\psi_i \ge 0$ everywhere. Natually, we define

$$q_i(x,y,t) = \int_\Omega p(x,y,t)\psi_i(y)dy \tag{1.11}$$

Because of the uniform bound on the L^1 norm of ψ_i and the sup-norm of φ_j on any compact set $K \subset \Omega$, we know that $<\psi_i,\varphi_j>_{L^2(\Omega)} \le Cj^{\frac{2k}{n}}$ for sufficiently large j with C depending on K and not on i,j. Hence

$$q_i(x,t) = \sum_{j=1}^{\infty} <\psi_i,\varphi_j>_{L^2(\Omega)} e^{-\lambda_j t}\varphi_j(x) \to p(x,y,t) \tag{1.12}$$

uniformly on compact set $K \subset \Omega$.

Now, we prove the following Weyl's asymptotifc formula for eigenvalues λ_j for the laplacian $-\triangle$.

Theorem 1.8. *For $\lambda \in \mathbb{R}$, let N_λ denote the number of eigenvalues λ_j of $-\triangle$ relative to Dirichlet boundary conditions which are $\le \lambda$, then*

$$N_\lambda \sim \frac{\lambda^{\frac{n}{2}}|\Omega|}{(4\pi)^{\frac{n}{2}}\Gamma(n/2+1)} \tag{1.13}$$

In order to prove the theorem, we need first to get some asymptotic estimates of the heat kernel on Ω, especially on the diagonal, then we will apply the Tauberian theorem to conclude the proof.

First, we consider the heat kernel of $\mathbb{R}^n$, denoted by

$$K(x,y,t) := \frac{e^{-\frac{|x-y|^2}{4t}}}{4\pi t^{n/2}} \tag{1.14}$$

and just like the case of $p(x,y,t)$, we can construct a sequence of $k_i(x,t) = \int_{\mathbb{R}^n} K(x,y,t)\psi_i(y)dy$. Our goal then is to show that when $t \to 0$, $p(x,y,t)$ is well approximated by $K(x,y,t)$, and the proof will use the following comparison:

Lemma 1.9. *(Parabolic Maximum Principle) Suppose Ω is a bounded domain, $u \in C^0\overline{(\Omega \times (0,T))} \cap C^2(\Omega \times (0,T))$, and*

$$u_t - \triangle u \leq 0 \tag{1.15}$$

in $\Omega \times (0,T)$ with $T > 0$. Then for each $t \leq T$,

$$sup_{\Omega\times(0,t)} u = sup_{(\partial\Omega\times(0,t))\cup(\Omega\times 0)} u \tag{1.16}$$

Proof. (Weyl's asymptotic formula) Based on the definition of the q_i and k_i, we see that $q_i(0) = k_i(0) = \psi_(x)$ on Ω and $q_i = 0 \leq k_i$ on $\partial\Omega \times [0,\infty)$, hence by Maximum Principle to , we know that

$$q_i(x,t) \leq k_i(x,t) \quad \text{for all } (x,t) \in \Omega \times (0,\infty) \tag{1.17}$$

and

$$sup_{(x,t)\in\overline{\Omega\times(0,\infty)}} k_i(x,t) - q_i(x,t) \leq sup_{(x,t)\in\partial\Omega\times(0,\infty)} k_i(x,t) \tag{1.18}$$

Then if we let $t \to 0^+$, then we get the following inequalities for fixed y

$$p(x,y,t) \leq K(x,y,t) \quad \text{for all } (x,t) \in \Omega \times (0,\infty) \tag{1.19}$$

and

$$sup_{(x,t)\in\overline{\Omega\times(0,\infty)}} K(x,y,t) - p(x,y,t) \leq sup_{(x,t)\in\partial\Omega\times(0,\infty)} K(x,y,t) \tag{1.20}$$

Now, since it is the $\int_\Omega p(x,x,t)dx = \sum_j e^{-\lambda_j t}$ that we are interested in, we look at the following set $\Omega_\sigma := \{x \in \Omega | \text{dist}(x,\partial\Omega) > \sigma\}$.Then it is easy to see that for $y \in \Omega_\sigma$

$$0 \leq K(y,y,t) - p(y,y,t) \leq sup_{t\in(0,\infty)} \frac{e^{-\frac{\sigma^2}{4t}}}{4\pi t^{n/2}} \sim C(n)\sigma^{-n} \tag{1.21}$$

and

$$0 \leq p(y,y,t) \leq K(y,y,t) = \frac{1}{4\pi t^{n/2}} \tag{1.22}$$

integrate over Ω_σ and Ω respective and combine those inequalities, we get

$$\lim_{t\to 0} \frac{1}{4\pi t^{n/2}} \sum_{j=1}^{\infty} e^{-\lambda_j t} = |\Omega| \tag{1.23}$$

By the Tauberian Theorem, we get our desired result. For reference, please check Feller Vol. 2, p.443, Th. 1.

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