MATH 742 THE MOSER-HARNACK INEQUALITY

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1. Some Preliminary Results

In this note, we will present a proof of Moser-Harnack inequality for weak solution u of the following homogeneous second-order elliptic PDE:

$$Lu = \sum_{i,j=1}^{n} D_j(a^{ij}(x)D_i u(x)) = 0 \tag{1.1}$$

with $\|a^{ij}\|_\infty < \infty$ and boundedness and uniform ellipticity condition on L, i.e. there exist $0 < \lambda \leq \Lambda < \infty$ such that

$$\lambda|\xi|^2 \leq a^{ij}\xi_i\xi_j \leq \Lambda|\xi|^2 \tag{1.2}$$

for all ξ in a bounded domain $\Omega \subset \mathbb{R}^n$.

Definition 1.1. A function $u \in W^{1,2}(\Omega)$ is called a wek subsolution of L if for all $\psi \in H^1_0(\Omega) \geq 0$ *a.e.*,

$$\int_\Omega a^{ij} D_i u D_i \psi \, dx \leq 0 \tag{1.3}$$

in this case, we write $Lu \geq 0$. Similarly if we can define a supersolution if we have $Lu \leq 0$.

Intuitively, we should think of u being convex and concave respectively.

Lemma 1.2. *Let u be a classical solution, and $f \in C^2(\mathbb{R})$ convex, then $f \circ u$ is a subsolution.*

Proof. $L(f \circ u) = D_j(a^{ij} f'(u) D_i) = f'' a^{ij} D_i u D_j u + f' Lu$ □

Now, we extend the above result to weak subsolutions. Here, we need to assume that the sup-norm of $|f'|$ and $|f''|$ are bounded, so we have $D_i(f \circ u) = f'(u) D_i u$ and $D_i(f' \circ u) = f''(u) D_i u$ and

$$\int_\Omega a^{ij} D_i(f \circ u) D_j \psi = \int_\Omega a^{ij} D_i u D_j(f'(u)\psi) - \int_\Omega a^{ij} D_i u D_j u f''(u)\psi \tag{1.4}$$

we see that if $f \in C^2(\Omega)$ is convex with $f' \geq 0$ then $f \circ u$ is a weak subsolution if u is a weak solution.

Lemma 1.3. *Let $u \in W^{1,2}(\Omega)$ be a weak subsolution of L, and $k \in \mathbb{R}$. Then $v := max(u(x), k)$ is a weak subsolution.*

Now we can state the main result of this note

Theorem 1.4. *Let u be a subsolution in the ball $B(x_0, R) \subset \mathbb{R}^n$ and $p > 2$, then*

$$\sup_{B(x_0,\theta R)} u \leq c_1 \Big(\frac{1}{|B(x_0, R)|} \int_{B(x_0,R)} (u^+)^p \, dx\Big)^{1/p} \tag{1.5}$$

where $u^+ := \max\{u(x), 0\}$ and c_1 only depends on λ, Λ and n, θ, p.
Similarly, we have

Theorem 1.5. *Let u be a positive supersolution in $B(x_0, R)$ and there exists $0 < p_0 < \frac{n}{n-2}$ with $n \geq 3$ such that*

$$\Big(\frac{1}{|B(x_0, R)|} \int_{B(x_0,R)} u^{p_0} \, dx\Big)^{1/p_0} \leq c_2 \inf_{B(x_0,\theta R)} u \tag{1.6}$$

with c_2 depends only on λ, Λ and n.

Remark 1.6. Take $u := min\{|x|^{2-n}, k\}$ for any given k, and take $L = \Delta$ in $\mathbb{R}^n$, since $\Delta u = 0$ on $\mathbb{R}^n - \{0\}$, it is a weak supersolution, but we see clearly that as k blows up, the $n/(n-2)$-norm of u also goes to infinity. Hence the condition that $p < n/(n-2)$ is necessary.

Now combining these two theorems and we get the following Harnack type result:

Corollary 1.7. Let u be a postivie weak solution of $Lu = 0$ in the ball $B(x_0, 4R)$,

$$\sup_{B(x_0,R)} u \leq c_3 \inf_{B(x_0,R)} u \tag{1.7}$$

with c_3 depends only on λ, Λ and n.

Now if we apply a standard chain argument we get the following more general result for any domain Ω

Corollary 1.8. Let u be a positive solution to $Lu = 0$ in the domain Ω, then for any compact sub-domain $\Omega_0 \subset \Omega$, we have

$$\sup_{\Omega_0} \leq c_3 \inf_{\Omega_0} u \tag{1.8}$$

with c_3 depends only on λ, Λ, n and $\Omega_0 \subset \Omega$.

Before we proceed to the proof of our theorems, let's state two useful lemmas first.

Lemma 1.9.

$$\lim_{p\to\infty} \Big(\fint_{B(x_0,R)} u^p \, dx\Big)^{1/p} = \sup_{B(x_0,R)} u \tag{1.9}$$

and

$$\lim_{p\to-\infty} \Big(\fint_{B(x_0,R)} u^p \, dx\Big)^{1/p} = \inf_{B(x_0,R)} u \tag{1.10}$$

Proof. An direct application of Holder equality and the definition of essential supremum. □

Lemma 1.10. *Let u be a positive subsolution in Ω, and for $q > 1/2$, set $v := u^q \in L^2(\Omega)$, for any $\eta \in H_0^1(\Omega)$, we have*

$$\int_\Omega \eta^2 |Dv|^2 \leq \frac{\Lambda^2}{\lambda^2} \Big(\frac{2q}{2q-1}\Big)^2 \int_\Omega |D\eta|^2 v^2 \tag{1.11}$$

Proof. Take $f(u) := u^{2q}$, clearly if u is a weak subsolution, so is $f(u)$ by Lemma 1.2. By take the following cleverly chosen test function $\psi := f'(u)\eta^2$, we have

$$\int_\Omega a^{ij} D_i u D_j \psi = \int_\Omega 2|q|(2q-1)u^{2q-2}\eta^2 a^{ij} D_i u D_j u + \int_\Omega 4|q|u^{2q-1}\eta a^{ij} D_i u D_j \eta \leq 0 \tag{1.12}$$

By uniform ellipticity of a^{ij} and Young's inequality to the last term above,

$$2|q|(2q-1)\lambda \int |Du|^2 u^{2q-2}\eta^2 \leq 2|q|\Lambda\epsilon \int |Du|^2 u^{2q-2}\eta^2 + \frac{2|q|\Lambda}{\epsilon}\int u^{2q}|D\eta|^2 \tag{1.13}$$

Pick $\epsilon = \frac{2q-1}{2}\frac{\lambda}{\Lambda}$, we get the desired result. □

We are now in the position to prove Theorem 1.4.

Remark 1.11. The theorem is true even if u is not bounded. However, for the sake of simplicity, we assume that u is essentially bounded. Note that the inequality in the theorem is scaling invariant, hence we could assume that $R = 1$ and $x_0 = 0$. Also, we can assume that u is positive, otherwise we look at $v_k = max\{k, u\}$ and then use a sequence argument for $k \to 0$, or directly look at u^+.

Proof. First, assume $p \geq 2$. For any test function η, we have

$$\int_{B_1} a^{ij} D_i u D_j \eta \leq 0 \tag{1.14}$$

Let $v := \eta^2 u^{p-1}$ with $\eta \in C_0^\infty(B_R)$, we have

$$(p-1)\int_{B_R} a^{ij} D_i u D_j u u^{p-2}\eta^2\, dx \leq -2\int_{B_R} a^{ij} u^{p-1}\eta D_i u D_j \eta\, dx \tag{1.15}$$

Again, by uniform ellipticity of a^{ij} and Young's inequality, there is a constant C such that

$$\int_{B_R} |D(\eta u^{p/2})|^2\, dx \leq C\int_{B_R} |D\eta|^2 u^p\, dx \tag{1.16}$$

Using Sobolev inequality, we get

$$\left(\int_{B_R} (\eta u^{p/2})^{2^*}\right)^{\frac{2}{2^*}} \leq C\int_{B_R} |D\eta|^2 u^p\, dx \tag{1.17}$$

where $2^* = 2n/(n-2)$.

Let $R_k = R(\theta + \frac{1-\theta}{2^k})$ with $\eta_k \in C_0^\infty(B_{R_k})$ such that

$$\begin{cases} \eta_k = 1 \text{ on } B_{R_{k+1}} \\ \eta = 0 \text{ on } \mathbb{R}^n - B_{R_k} \\ |D\eta_k| \leq \frac{2^{k+1}}{(1-\theta)R} \end{cases} \tag{1.18}$$

In a more compact way, the above inequality can then be expressed as the following

$$||u||_{\frac{np}{n-2}}(B_{R_{k+1}}) \leq (C4^k)^{1/p}||u||_p(B_{R_k}) \tag{1.19}$$

Take $p_k = (\frac{n}{n-2})^k p$, we thus have

$$||u||_{p_{k+1}}(B_{R_{k+1}}) \leq (C4^k)^{1/p_k}||u||_{p_k}(B_{R_k}) \tag{1.20}$$

Since

$$log(\prod_{k=1}^\infty (C4^k)^{\frac{1}{p_k}}) < C \tag{1.21}$$

We can then take $k \to \infty$ and get

$$\|u\|_\infty(B_{\theta R}) \le \frac{C}{(R-\theta R)^{n/p}}\frac{1}{R^n}\|u\|_p(B_R) \tag{1.22}$$

Hence we get our desired inequality in Theorem 1.1.

□

We comment that the proof of Theorem 1.2 is similar in spirit with more complicated iteration procedures, hence we omit here and refer the interested reader to Gilbarg and Trudinger for more comprehensive treatment.

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