Bochner Technique:

Most of this heavily references Peter Petersen's Riemannian Geometry book. [Left to put in: Proof of Killing's Equation, Relationship of Lie algebra of Killing fields to Lie algebra of the isometry group of M]

A vector field X is Killing if the local flows generated by X act by isometries. We will prove the following theorem:

Theorem 1.5: (Bochner, 1946) *Suppose (M, g) is compact, oriented, and has Ric ≤ 0. Then every Killing field is parallel. Furthermore, if Ric < 0, then there are no nontrivial Killing fields.*

The theorem is important because it constrains the isometry group of (M, g). For instance if Ric < 0, then the isometry group of M is finite.

Now suppose X is a Killing field. Let $f = \frac{1}{2}|X|^2 = \frac{1}{2}\langle X, X\rangle$. We would like to produce the following formula which will help us prove the theorem:

$$\Delta f = -\mathrm{Ric}\,(X, X) + |\nabla X|^2\,. \tag{1}$$

Some Explanation of the formula:

1.) Here Ric is the Ricci Curvature, which is the metric contraction of the Curvature tensor R in the 1 and 4 places: (if $\{e_i\}$ is an orthonormal basis of T_pM)

$$\mathrm{Ric}\,(V, W) = \sum_i \langle R\,(e_i, V)\,W, e_i\rangle\,.$$

2.) $|\nabla X|$ is the Euclidean norm of the $(1,1)$-tensor ∇X, which we view as a linear endomorphism $\nabla X : TM \to TM$, given by

$$(\nabla X)\,(v) = \nabla_v X.$$

In coordinates $\nabla X = \frac{\partial}{\partial x^i} X^j E_j \otimes \sigma^i + X^j \Gamma^k_{ij} \cdot E_k \otimes \sigma^i$.

Brief Review of the Euclidean norm:

Let T be a $(1,1)$-tensor which we interpret as an endomorphism $T : TM \to TM$ and is given in coordinates by

$$T^i_j \cdot E_i \otimes \sigma^j.$$

In general the Euclidean norm of T is given by

$$|T| = \sqrt{\mathrm{tr}\,(T \circ T^*)}$$

where $T^* : TM \to TM$ is the adjoint of T (here interpreted after type change using the metric g). [T^* would be a map from $T^*M \to T^*M$ of the form

$$T^j_i \cdot \sigma^i \otimes E_j,$$

but since T_pM is an inner product space w.r.t. g, we can make the identifications

$$\sigma^i \mapsto g^{ij} E_j$$

$$E_j \mapsto g_{ji} \sigma^i$$

which converts T^* to a map from $TM \to TM$:

$$T^j_i g_{jk} g^{il} \cdot E_l \otimes \sigma^k.]$$

So

$$\begin{aligned}
(T \circ T^*)(E_t) &= (T^r_s \cdot E_r \otimes \sigma^s)\left(\left(T^j_i g_{jk} g^{il} \cdot E_l \otimes \sigma^k\right)(E_t)\right) \\
&= (T^r_s \cdot E_r \otimes \sigma^s)\left(T^j_i g_{jt} g^{il} \cdot E_l\right) \\
&= T^r_l T^j_i g_{jt} g^{il} \cdot E_r
\end{aligned}$$

which means that

$$(T \circ T^*) = T^r_l T^j_i g_{jt} g^{il} \cdot E_r \otimes \sigma^t$$

so that the trace is just

$$\mathrm{tr}\,(T \circ T^*) = T^r_l T^j_i g_{jr} g^{il}.$$

3.) Δf is the Laplacian of f:

$$\mathrm{div}\,(\mathrm{grad}\ f)$$

where $\mathrm{grad} f$ is the vector field defined such that

$$\langle \mathrm{grad}\ f, V \rangle = V(f) = D_V f$$

for all vector fields V, and div X is the trace of the linear map $Y \mapsto \nabla_Y X$. In coordinates this map is given by

$$\frac{\partial}{\partial x^j}\left(X^i\right) E_i \otimes \sigma^j + X^i \Gamma^k_{ji} \cdot E_k \otimes \sigma^j,$$

and the trace is given by $\frac{\partial}{\partial x^i}\left(X^i\right) + X^i \Gamma^j_{ji}$.

For the following we need Killing's equation: If X is a Killing field on M then

$$\langle \nabla_Y X, Z\rangle + \langle \nabla_Z X, Y\rangle = 0.$$

We first define a skew-adjoint (1,1)-tensor by $T(v) = \nabla_v X$. To be skew-adjoint means that $\forall\, v, w \in T_pM$:

$$\langle T(v), w\rangle = -\langle v, T(w)\rangle \quad \text{or equiv.} \quad \langle T(v), v\rangle = 0$$

To show Formula (1) we prove the following in sequence:

(1) grad $f = \nabla f = -T(X) = -\nabla_X X$:

$$\begin{aligned}
\langle \text{grad} f, V\rangle &= V(f) \\
&= \frac{1}{2} V\langle X, X\rangle \\
&= \frac{1}{2}\left(\langle \nabla_V X, X\rangle + \langle X, \nabla_V X\rangle\right) \\
&= \langle \nabla_V X, X\rangle \\
&= -\langle \nabla_X X, V\rangle .
\end{aligned}$$

(2) $\nabla^2 f = \nabla(\text{grad } f) = -T^2 - \nabla_X T - R_X$ (where $R_X(V) = R(V, X)X$):

Apply $\nabla^2 f$ to a vector field V:

$$\begin{aligned}
\left(\nabla^2 f\right)(V) &= \nabla_V(-\nabla_X X) \\
&= -R(V, X)X - \nabla_X \nabla_V X - \nabla_{[V,X]} X
\end{aligned}$$

which comes from the definition of the curvature tensor. This equals

$$= -R_X(V) - \nabla_X \nabla_V X + \nabla_{\nabla_X V} X - \nabla_{\nabla_V X} X$$

because $[V, X] = \nabla_V X - \nabla_X V$, since ∇ is symmetric. Since $T \circ T(V) = T(\nabla_V X) = \nabla_{\nabla_V X} X$, this equals

$$= -R_X(V) - T \circ T(V) - \nabla_X \nabla_V X + \nabla_{\nabla_X V} X.$$

Leibnitz Rule for covariant derivatives of tensors:

Because we require that

$$\nabla_X(T(V)) = (\nabla_X T)(V) + T(\nabla_X V)$$

we have that

$$(\nabla_X T)(V) = \nabla_X \nabla_V X - \nabla_{\nabla_X V} X$$

so

$$\left(\nabla^2 f\right)(V) = -(R_X)(V) - \left(T^2\right)(V) - (\nabla_X T)(V)$$

which shows (2).

(3) We take the trace of $\nabla^2 f = -T^2 - \nabla_X T - R_X$, to get
$\Delta f = -\text{Ric}(X, X) + |T|^2 = -\text{Ric}(X, X) + |\nabla X|^2$:

This is because of 3 facts:

For skew symmetric (1,1)-tensors (in coordinates $T = T^i_j \cdot E_i \otimes \sigma_j$):

(a) $T^* = -T$. Which implies that $\mathrm{tr}\left(-T^2\right) = \mathrm{tr}\left(T \circ T^*\right) = |T|^2$.
(b) $\mathrm{tr}\left(T\right) := T^i_i = 0$.
(c) The covariant derivative of T (in the direction of a vector field X) is also skew symmetric.

Proof of (b):

Let T be a $(1,1)$-tensor that is skew-symmetric w.r.t. the metric g

$$\langle T\left(v\right), v\rangle = 0 \quad \forall v,$$

with components T^i_j written w.r.t. a frame and dual frame $\{E_i\}$ and $\left\{\sigma^i\right\}$. We want to first transform T so that it is w.r.t. an orthonormal basis $\left\{\bar{E}_i\right\}$ w.r.t. g. Let A be such a coordinate tranformation matrix

$$\bar{E}_i = A^j_i E_j.$$

Then the transformed components of T are given by

$$\bar{T}^i_j = A^i_l T^l_k \left(A^{-1}\right)^k_j.$$

In our new frame it is of course still true that for any v

$$\left\langle \left(\bar{T}^i_j \cdot \bar{E}_i \otimes \bar{\sigma}^j\right)\left(v\right), v\right\rangle = 0.$$

Let $v = \bar{E}_k$ for a fixed k. Then

$$0 = \left\langle \left(\bar{T}^i_j \cdot \bar{E}_i \otimes \bar{\sigma}^j\right)\left(\bar{E}_k\right), \bar{E}_k\right\rangle = \bar{T}^k_k$$

where the last expression is not meant to be a summation, but just the kth diagonal element of the matrix $\bar{T}^i_j$. Since all diagonal elements of $\bar{T}^i_j$ are 0, the trace of $\left(\bar{T}^i_j \cdot \bar{E}_i \otimes \bar{\sigma}^j\right)$ is $\sum_k \bar{T}^k_k = 0$. Now $\bar{T}^i_j$ is related to T^i_j by a similarity transformation $A \Longrightarrow \mathrm{tr}\left(T^i_j\right) = 0$ also.

Proof of (c):

That T is skew symmetric means that for any vector field v, $\langle T\left(v\right), v\rangle \equiv 0$ on M. So for any vector field X:

$$X\left\langle T\left(v\right), v\right\rangle = 0.$$

Then

$$\begin{aligned} 0 = X\left\langle T\left(v\right), v\right\rangle &= \left\langle \nabla_X\left(T\left(v\right)\right), v\right\rangle + \left\langle T\left(v\right), \nabla_X v\right\rangle \\ &= \left\langle \left(\nabla_X T\right)\left(v\right) + T\left(\nabla_X v\right), v\right\rangle + \left\langle T\left(v\right), \nabla_X v\right\rangle \\ &= \left\langle \left(\nabla_X T\right)\left(v\right) + T\left(\nabla_X v\right), v\right\rangle - \left\langle v, T\left(\nabla_X v\right)\right\rangle \\ &= \left\langle \left(\nabla_X T\right)\left(v\right), v\right\rangle \end{aligned}$$

where we used the skew adjointness of T to get from the 2nd to 3rd line. So $\nabla_X T$ is skew symmetric also.

Before we go on we must have the following result which is proved using Stoke's theorem:

If M is a compact oriented manifold, and $d$vol is the volume form, then for any smooth function f

$$\int_M \Delta f \, d\text{vol} = 0.$$

We want to now use this result and our formula (1) to prove the theorem. We restate it:

Theorem 1.5: *Suppose (M, g) is compact, oriented, and has Ric ≤ 0. Then every Killing field is parallel. Furthermore, if Ric < 0, then there are no nontrivial Killing fields.*

Proof: Let X be a Killing Field and define $f = \frac{1}{2} |X|^2$. Since Ric ≤ 0

$$\begin{aligned} 0 &= \int_M \Delta f \, d\text{vol} \\ &= \int_M \left(-\text{Ric}(X, X) + |\nabla X|^2 \right) d\text{vol} \\ &\geq \int_M |\nabla X|^2 \, d\text{vol} \\ &\geq 0 \end{aligned}$$

so $|\nabla X| \equiv 0$, and X is parallel.

If in addition Ric < 0, then for $\text{Ric}(V, W)$ to equal 0, either V or W must be the 0 vector. This means that $\text{Ric}(X, X) \equiv 0$ iff $X \equiv 0$. So X must be a trivial vector field.

Ch. 6, L. Simon's Lectures in PDE:

We consider PDE's of the form

$$\sum_{|\alpha|,|\beta|\leq m} D^{\beta}\left(a_{\alpha\beta}D^{\alpha}u\right) = \sum_{|\beta|\leq m} D^{\beta}f_{\beta} \tag{2}$$

where f_{β} are prescribed $L^2_{\text{loc}}(\Omega)$ functions and $a_{\alpha\beta}$ are locally bounded functions, $a_{\alpha\beta}\in L^{\infty}_{\text{loc}}(\Omega)$.

u is a weak solution to (1) if when we multiply each side by $\zeta\in C^{\infty}_{c}(\Omega)$, a test function, and integrate we get equality:

$$\int_{\Omega}\sum_{|\alpha|,|\beta|\leq m}(-1)^{|\beta|}a_{\alpha\beta}D^{\alpha}uD^{\beta}\zeta = \int_{\Omega}\sum_{|\beta|\leq m}(-1)^{|\beta|}f_{\beta}D^{\beta}\zeta, \tag{3}$$

written out after integration by parts.

Here we only consider regularity results on a ball $B_R(x_0)$, where $\overline{B_R}(x_0)\subset\Omega$.

(E), an ellipticity condition: There exists a $\mu>0$ s.t.

$$\sum_{|\alpha|,|\beta|=m} a_{\alpha\beta}(x)\lambda_{\alpha}\lambda_{\beta}\geq\mu\sum_{|\alpha|=m}(\lambda_{\alpha})^2$$

for all $x\in\Omega$, and all collections of real numbers $\{\lambda_{\alpha}\}_{|\alpha|=m}$.

(B_k) $a_{\alpha\beta}\in W^{k,\infty}(\Omega)$ and there exists an $M>0$ s.t.

$$|D^{\gamma}a_{\alpha\beta}(x)|\leq M \text{ a.e. } x\in B_R(x_0),\ |\gamma|\leq k.$$

Our Main Theorem:

Theorem 1: *Assume that in* (1), $f_{\beta}\in H^k(\Omega)$. *If* $u\in H^m(B_R(x_0))$ *is a weak solution of* (1), *if* $k\geq 0$, *and if* (E) and (B_k) *hold, then* $u\in H^{m+k}_{loc}(B_R(x_0))$ *and*

$$\|u\|_{m+k,B_{\theta R}(x_0)}\leq C\left(\|u\|_{m-1,B_R(x_0)}+\sum_{|\beta|\leq m}\|f_{\beta}\|_{k,B_R(x_0)}\right)$$

for any choice of $\theta\in(0,1)$, *where* C *is a constant depending only on* n,m,k,θ,M,μ.

Note: Since the Sobolev embedding theorem can be used to show for l with $m+k>n/2+l$, that $H^{m+k}_{\text{loc}}(B_R(x_0))\subset C^l(\Omega)$ and

$$|u|_{C^l(B_R(x_0))}\leq C\|u\|_{m+k,B_R(x_0)},$$

(this is the *harder* version of the embedding theorem) [Here C depends on ???] we can show that under the conditions of Theorem 1, $u\in C^l(B_R(x_0))$.

We need to establish a helpful lemma:

Notation for Lemma:

(B) explicit boundedness of $a_{\alpha\beta}$:

$$|a_{\alpha\beta}(x)|\leq M,\quad \forall\, x\in B_R(x_0),\ |\alpha|,|\beta|\leq m.$$

Lemma 1: *If* $u\in H^m_{loc}(\Omega)$ *is a weak solution of* (1), *and if* (E) *and* (B) *hold, and if* $\overline{B_R}(x_0)\subset\Omega$, *then for each* $\theta\in(0,1)$ *we have*

$$\|u\|_{m,B_{\theta R}(x_0)}\leq C\left(\|u\|_{m-1,B_R(x_0)}+\sum_{|\beta|\leq m}\|f_{\beta}\|_{k,B_R(x_0)}\right),$$

where C *depends only on* R,M,μ,m,n,θ.

To prove Lemma 1 we need another lemma (also Lemma 1, but from section 5 of Simon's PDE's).

Pre-Lemma 1: If $u \in W^{m,p}_{\text{loc}}(\Omega)$ amd if $|\alpha| \le m$, then $D^\alpha u_\sigma \to D^\alpha u$ pointwise a.e. in Ω, and also locally w.r.t. the $\|\cdot\|_{m,p}$ norm in Ω.

(Here u_σ is a mollification of u with respect to a sequence of mollifiers ρ_σ.)

Proof of Lemma 1:

We have to first show that if u is a weak solution to (1) then (2) will also be satisfied when $\zeta = \varphi h$ with $\varphi \in C^\infty_c(\Omega)$ and $h \in H^m_{\text{loc}}(\Omega)$. This is true because $\varphi h_\sigma \in C^\infty_c(\Omega)$ for sufficiently small σ and, by the pre-lemma, $\lim_{\sigma\to 0} \varphi h_\sigma = \sigma h$ w.r.t. the $H^m(\Omega)$ norm: It will be true that

$$\int_\Omega \sum_{|\alpha|,|\beta|\le m} (-1)^{|\beta|} a_{\alpha\beta} D^\alpha u D^\beta(\varphi h_\sigma) = \int_\Omega \sum_{|\beta|\le m} (-1)^{|\beta|} f_\beta D^\beta(\varphi h_\sigma)$$

by definition of weak solution, and since

$$\left(\sum_{|\beta|\le m} \int \left|D^\beta(\varphi h_\sigma) - D^\beta(\varphi h)\right|^2\right)^{1/2} \to 0 \quad \text{as} \quad \sigma \to 0$$

we get

$$\int_\Omega \sum_{|\alpha|,|\beta|\le m} (-1)^{|\beta|} a_{\alpha\beta} D^\alpha u D^\beta(\varphi h) = \int_\Omega \sum_{|\beta|\le m} (-1)^{|\beta|} f_\beta D^\beta(\varphi h).$$

Now make the careful choice $h = u$, and we get

$$\int_\Omega \sum_{|\alpha|,|\beta|\le m} (-1)^{|\beta|} a_{\alpha\beta} D^\alpha u \left(\sum_{\gamma+\delta=\beta} \frac{\beta!}{\gamma!\delta!} D^\gamma u D^\delta \varphi\right) = \int_\Omega \sum_{|\beta|\le m} (-1)^{|\beta|} f_\beta \left(\sum_{\gamma+\delta=\beta} \frac{\beta!}{\gamma!\delta!} D^\gamma u D^\delta \varphi\right).$$

We now impose (E) and (B) to get (3/4):

$$\int_\Omega \sum_{|\alpha|=m} |D^\alpha u|^2 \varphi \le m! \int_\Omega \sum_{|\beta|\le m} \left(|f_\beta| \sum_{\gamma+\delta=\beta} \left(|D^\gamma u| \left|D^\delta \varphi\right|\right)\right) \tag{4}$$

$$+ Mm! \int_\Omega \left(\sum_{|\alpha|\le m} |D^\alpha u|\right) \left(\sum_{|\delta|\le m, |\gamma|\le m-1} \left(|D^\gamma u| \left|D^\delta \varphi\right|\right)\right) \tag{5}$$

Details:
Set $D^{\alpha}u = \lambda_{\alpha}$ in (E), and multiply both sides by φ (which is positive) gives

$$\mu \int_{\Omega} \sum_{|\alpha|=m} (D^{\alpha}u)^2 \varphi \leq \int_{\Omega} \sum_{|\alpha|,|\beta|=m} a_{\alpha\beta} (D^{\alpha}u) \left(D^{\beta}u\right) \varphi.$$

Now adding and subtracting some equal terms gives

$$\begin{aligned}
\int_{\Omega} \sum_{|\alpha|,|\beta|=m} a_{\alpha\beta} (D^{\alpha}u) \left(D^{\beta}u\right) \varphi \;&=\; \int_{\Omega} \sum_{|\alpha|,|\beta|=m} a_{\alpha\beta} (D^{\alpha}u) \left(D^{\beta}u\right) \varphi \\
&\quad + \int_{\Omega} \sum_{|\alpha|,|\beta|=m} \left((-1)^{|\beta|} a_{\alpha\beta} D^{\alpha}u \left(\sum_{\gamma+\delta=\beta, |\delta|\geq 1} \frac{\beta!}{\gamma!\delta!} D^{\gamma}u D^{\delta}\varphi \right) \right) \\
&\quad + \int_{\Omega} \sum_{|\alpha|<m \text{ or } |\beta|<m} \left((-1)^{|\beta|} a_{\alpha\beta} D^{\alpha}u \left(\sum_{\gamma+\delta=\beta} \frac{\beta!}{\gamma!\delta!} D^{\gamma}u D^{\delta}\varphi \right) \right) \\
&\quad - \int_{\Omega} \sum_{|\alpha|,|\beta|=m} \left((-1)^{|\beta|} a_{\alpha\beta} D^{\alpha}u \left(\sum_{\gamma+\delta=\beta, |\delta|\geq 1} \frac{\beta!}{\gamma!\delta!} D^{\gamma}u D^{\delta}\varphi \right) \right) \\
&\quad - \int_{\Omega} \sum_{|\alpha|<m \text{ or } |\beta|<m} \left((-1)^{|\beta|} a_{\alpha\beta} D^{\alpha}u \left(\sum_{\gamma+\delta=\beta} \frac{\beta!}{\gamma!\delta!} D^{\gamma}u D^{\delta}\varphi \right) \right) \\
&= \int_{\Omega} \sum_{|\alpha|,|\beta|\leq m} \left((-1)^{|\beta|} a_{\alpha\beta} D^{\alpha}u \left(\sum_{\gamma+\delta=\beta} \frac{\beta!}{\gamma!\delta!} D^{\gamma}u D^{\delta}\varphi \right) \right) \\
&\quad - \int_{\Omega} \sum_{|\alpha|,|\beta|=m} \left((-1)^{|\beta|} a_{\alpha\beta} D^{\alpha}u \left(\sum_{\gamma+\delta=\beta, |\delta|\geq 1} \frac{\beta!}{\gamma!\delta!} D^{\gamma}u D^{\delta}\varphi \right) \right) \\
&\quad - \int_{\Omega} \sum_{|\alpha|<m \text{ or } |\beta|<m} \left((-1)^{|\beta|} a_{\alpha\beta} D^{\alpha}u \left(\sum_{\gamma+\delta=\beta} \frac{\beta!}{\gamma!\delta!} D^{\gamma}u D^{\delta}\varphi \right) \right) \\
&= \int_{\Omega} \sum_{|\beta|\leq m} \left((-1)^{|\beta|} f_{\beta} \sum_{\gamma+\delta=\beta} \left(\frac{\beta!}{\gamma!\delta!} D^{\gamma}u D^{\delta}\varphi \right) \right) \\
&\quad - \text{the two negative terms in the previous line.}
\end{aligned}$$

Taking absolute values and distributing across integrals and summations gives (3)

Let $\theta \in (0,1)$, and choose φ to be a cutoff function with for the balls $B_{\theta R}(x_0)$ and $B_R(x_0)$ in the usual sense:

$$\varphi \in C_c^{\infty}(\Omega), 0 \leq \varphi \leq 1,\ \varphi \equiv 0 \text{ outside of } B_R(x_0)$$

$$\varphi \equiv 1 \text{ on } B_{\theta R}(x_0),\ \sup \left|D^{\delta}\varphi\right| \leq C\left((1-\theta) R\right)^{-|\delta|}$$

for any multi-index δ with $|\delta| \leq m$ [Our proof doesn't depend on the case where $|\delta| > m$]. Also C can be made to only depend on δ.

With such a φ we have that

$$\sup \left|D^{\delta}\varphi^{2m}\right| \leq C\left((1-\theta) R\right)^{-|\delta|} \varphi^{m}, \quad \text{for any } \delta \text{ with } |\delta| \leq m. \tag{6}$$

Details (by induction):
This comes down to an appropriate use of the chain rule for weak derivatives:

$$\sup \left|D^\delta \varphi^2\right| \leq \sup \left|2\varphi D^\delta \varphi\right|.$$

So

$$D^\delta \varphi^{2(m+1)} = \sum_{\alpha+\beta=\delta} \frac{\delta!}{\alpha!\beta!} D^\alpha \left(\varphi^{2m}\right) D^\beta \left(\varphi^2\right)$$

so

$$\begin{aligned}\sup \left|D^\delta \varphi^{2(m+1)}\right| &\leq \sum_{\alpha+\beta=\delta} C\left(\left(1-\theta\right)R\right)^{-|\alpha|} \varphi^m \cdot 2\varphi C \left(\left(1-\theta\right)R\right)^{-|\beta|} \varphi \\ &\leq C' \left(\left(1-\theta\right)R\right)^{-|\delta|} \varphi^{m+1}\end{aligned}$$

for some new constant C'.

Now (3) will also hold for the $C_c^\infty(\Omega)$ function $\varphi' = \varphi^{2m}$. If we substitute then φ^{2m} in place of φ in (3/4), and use (5) twice we get:

$$\begin{aligned}\text{new } LHS &= \int_{B_R(x_0)} \sum_{|\alpha|=m} |D^\alpha u| \varphi^{2m} \\ &\leq m! \int_{B_R(x_0)} \sum_{|\beta|\leq m} \left(|f_\beta| \sum_{\gamma+\delta=\beta} \left(|D^\gamma u| \left|D^\delta \varphi^{2m}\right|\right)\right) \\ &\quad + Mm! \int_{B_R(x_0)} \left(\sum_{|\alpha|\leq m} |D^\alpha u|\right) \left(\sum_{|\delta|\leq m, |\gamma|\leq m-1} \left(|D^\gamma u| \left|D^\delta \varphi^{2m}\right|\right)\right) \\ &\leq m! \int_{B_R(x_0)} \left(\sum_{|\alpha|\leq m} |D^\alpha u|\right) \left(\sum_{|\beta|\leq m} |f_\beta| \varphi^m \cdot \sup_{|\delta|\leq m} \left\{C\left(\left(1-\theta\right)R\right)^{-|\delta|}\right\}\right) \\ &\quad + Mm! \int_{B_R(x_0)} \left(\sum_{|\alpha|\leq m} |D^\alpha u|\right) \left(\sum_{|\gamma|\leq m-1} \left(|D^\gamma u| \varphi^m\right) \cdot \sup_{|\delta|\leq m} \left\{C\left(\left(1-\theta\right)R\right)^{-|\delta|}\right\}\right) \\ &\leq C \int_{B_R(x_0)} \left(\sum_{|\alpha|\leq m} |D^\alpha u| \varphi^m\right) \left(\sum_{|\gamma|\leq m-1} |D^\gamma u| + \sum_{|\beta|\leq m} |f_\beta|\right)\end{aligned}$$

for some new constant C which depends on M, μ, θ, R, m, n.

Rewrite the last line as

$$C \int_{B_R(x_0)} \left(\sum_{|\alpha|=m} |D^\alpha u| \varphi^m\right) \left(\sum_{|\gamma|\leq m-1} |D^\gamma u| + \sum_{|\beta|\leq m} |f_\beta|\right) \tag{7}$$

$$+C \int_{B_R(x_0)} \left(\sum_{|\alpha|\leq m-1} |D^\alpha u| \varphi^m\right) \left(\sum_{|\gamma|\leq m-1} |D^\gamma u| + \sum_{|\beta|\leq m} |f_\beta|\right) \tag{8}$$

Term (6) is the same as

$$C \int_{B_R(x_0)} \left(\sum_{|\alpha|=m, |\gamma|\leq m-1} |D^\alpha u| \varphi^m |D^\gamma u|\right) + C \int_{B_R(x_0)} \left(\sum_{|\alpha|=m, |\beta|\leq m} |D^\alpha u| \varphi^m |f_\beta|\right)$$

Now use Cauchy's inequality $ab \le \varepsilon a^2/2 + b^2/(2\varepsilon)$ on these terms, to show that they are less than

$$2CK\varepsilon \int_{B_R(x_0)} \sum_{|\alpha|=m} |D^\alpha u|^2 \varphi^{2m} + \frac{CK}{2\varepsilon} \int_{B_R(x_0)} \left(\sum_{|\gamma|\le m-1} |D^\gamma u|^2 + \sum_{|\beta|\le m} |f_\beta|^2 \right)$$

where $K =$ number of multi-indices $\le m$. Do the same for term (7) to get that it is $\le$:

$$2CK\varepsilon \int_{B_R(x_0)} \sum_{|\alpha|\le m-1} |D^\alpha u|^2 \varphi^{2m} + \frac{CK}{2\varepsilon} \int_{B_R(x_0)} \left(\sum_{|\gamma|\le m-1} |D^\gamma u|^2 + \sum_{|\beta|\le m} |f_\beta|^2 \right)$$

Use this to write

$$\begin{aligned} \int_\Omega \sum_{|\alpha|=m} |D^\alpha u|^2 \varphi^{2m} \quad \le \quad & 2CK\varepsilon \int_{B_R(x_0)} \sum_{|\alpha|=m} |D^\alpha u|^2 \varphi^{2m} + \frac{CK}{2\varepsilon} \int_{B_R(x_0)} \left(\sum_{|\gamma|\le m-1} |D^\gamma u|^2 + \sum_{|\beta|\le m} |f_\beta|^2 \right) \\ & + 2CK\varepsilon \int_{B_R(x_0)} \left(\sum_{|\alpha|\le m-1} |D^\alpha u|^2 \varphi^{2m} \right) + \frac{CK}{2\varepsilon} \int_{B_R(x_0)} \left(\sum_{|\gamma|\le m-1} |D^\gamma u|^2 + \sum_{|\beta|\le m} |f_\beta|^2 \right). \end{aligned}$$

We can pull over the 1st and 3rd term on the RHS to the other side, and add in to both sides

$$+ \int_{B_R(x_0)} \left(\sum_{|\alpha|\le m-1} |D^\alpha u|^2 \varphi^{2m} \right)$$

to get

$$(1 - 2CK\varepsilon) \int_\Omega \sum_{|\alpha|\le m} |D^\alpha u|^2 \varphi^{2m} \le \left(\frac{CK}{\varepsilon} + 1 \right) \int_{B_R(x_0)} \left(\sum_{|\gamma|\le m-1} |D^\gamma u|^2 + \sum_{|\beta|\le m} |f_\beta|^2 \right)$$

$\Longrightarrow$ (with appropriate choice of ε)

$$\int_\Omega \sum_{|\alpha|\le m} |D^\alpha u|^2 \varphi^{2m} \le C' \int_{B_R(x_0)} \left(\sum_{|\gamma|\le m-1} |D^\gamma u|^2 + \sum_{|\beta|\le m} |f_\beta|^2 \right)$$

for some new constant C' depending on M, μ, θ, R, m, n. This implies Lemma 1.

For the proof of the Theorem we need some results about difference quotients:

If f is a real valued function defined on a domain $\Omega \subset \mathbb{R}^n$, a difference quotient in the j-th direction is

$$\Delta_j^h f := \frac{f(x + he_j) - f(x)}{h}$$

defined on a domain $\Omega_{|h|}$. We need the following 6 facts (for $h \ne 0$):

(a) $\Delta_j^h (f + g) = \Delta_j^h f + \Delta_j^h g$ on $B_{R-|h|}(x_0)$(Obvious)

(b) $\Delta_j^h (fg) = g\Delta_j^h f + \tilde{f}\Delta_j^h g$ for $f, g \in L^1(B_R(x_0))$, and where $\tilde{f}(x) = f(x + he_j)$:

$$\begin{aligned} \Delta_j^h (fg) &= \frac{1}{h}\left(f(x + he_j) g(x + he_j) - f(x) g(x)\right) \\ &= \frac{1}{h}\left(f(x + he_j) g(x + he_j) - f(x + he_j) g(x) + f(x + he_j) g(x) - f(x) g(x)\right) \\ &= g\Delta_j^h f + \tilde{f}\Delta_j^h \end{aligned}$$

(c) $D^\alpha \Delta_j^h f = \Delta_j^h D^\alpha f$ on $B_{R-|h|}(x_0)$ and for $f \in H^m(B_R(x_0))$ and $|\alpha| \leq m$: This follows from distributivity of the weak derivative.

(d) Integration by parts formula:

$$\int_{B_R(x_0)} f \Delta_j^h g \, dx = -\int_{B_R(x_0)} g \Delta_j^{-h} f \, dx$$

whenver $f, g \in L^1(B_R(x_0))$ and fg vanishes outside of $B_{R-|h|}(x_0)$. By straight forward calculation:

$$\int_{B_R(x_0)} f \Delta_j^h g \, dx = \frac{1}{h} \int_{B_R(x_0)} f(x) g(x + he_j) - \frac{1}{h} \int_{B_R(x_0)} f(x) g(x)$$

and

$$\begin{aligned} -\int_{B_R(x_0)} g \Delta_j^{-h} f \, dx &= -\left[\left(\frac{1}{-h} \right) \int_{B_R(x_0)} g(x) f(x + he_j) - \left(\frac{1}{-h} \right) \int_{B_R(x_0)} g(x) f(x) \right] \\ &= \frac{1}{h} \int_{B_R(x_0)} g(y - he_j) f(y) - \frac{1}{h} \int_{B_R(x_0)} g(x) f(x) \end{aligned}$$

which are the same after change of variables $y = x + he_j$.

(e) Our condition (B_k) implies:

$$\left| \Delta_j^h D^\gamma a_{\alpha\beta}(x) \right| \leq M \quad \text{a.e.} \quad x \in \Omega_{|h|}, \ |\gamma| \leq k - 1.$$

(f) If $v \in H^l(B_R(x_0))$ then

$$\|\Delta_j^h v\|_{l-1, B_{R-|h|}(x_0)} \leq \|v\|_{l, B_R(x_0)}$$

Details of (e):
Because $|\gamma| < k$ there exists one more weak derivative and we can write

$$\Delta_j^h (D^\gamma a_{\alpha\beta}) = \frac{1}{h} \int_0^h D_j D^\gamma (x + te_j) \, dt.$$

Suppose (for purposes of getting a contradiction) that

$$\sup |D_j D^\gamma a_{\alpha\beta}| < \Delta_j^h (D^\gamma a_{\alpha\beta}).$$

Then

$$\begin{aligned} \Delta_j^h (D^\gamma a_{\alpha\beta}) &= \frac{1}{h} \int_0^h D_j D^\gamma (x + te_j) \, dt \\ &= D_j (D^\gamma a_{\alpha\beta}) \\ &< \Delta_j^h (D^\gamma a_{\alpha\beta}) \end{aligned}$$

which is a contradiction.

Details of (f):
We show that $\|\Delta_j^h v\|_{0,B_{R-|h|}(x_0)} \leq \|D_j v\|_{0,B_R(x_0)}$where $v \in H^1(B_R(x_0))$. This will show (f) by applying it to each term in the H^{l-1} norm:

$$\begin{aligned}
\left|\Delta_j^h v\right| = \left|\frac{v(x+he_j)-v(x)}{h}\right| &\leq \frac{1}{h}\int_0^h |D_j v(x_1,\ldots,x_j+\tau,\ldots x_n)|\,d\tau \\
&\leq \frac{1}{h}\cdot h^{1/2}\left(\int_0^h |D_j v(x_1,\ldots,x_j+\tau,\ldots x_n)|^2\,d\tau\right)^{1/2} \\
\Longrightarrow & \\
\left|\Delta_j^h v\right|^2 &\leq \frac{1}{h}\int_0^h |D_j v(x_1,\ldots,x_j+\tau,\ldots x_n)|^2\,d\tau \\
\Longrightarrow & \\
\int_{B_{R-|h|}(x_0)} \left|\Delta_j^h v\right|^2 &\leq \frac{1}{h}\int_{B_{R-|h|}(x_0)}\int_0^h |D_j v|^2\,d\tau dx \\
&\leq \frac{1}{h}\int_0^h\int_{B_{R-|h|}(x_0)} |D_j v|^2\,dx d\tau \\
&= \frac{1}{h}\int_0^h \|D_j v\|_{0,B_{R-|h|}(x_0)} d\tau \\
&= \|D_j v\|_{0,B_{R-|h|}(x_0)} \leq \|D_j v\|_{0,B_R(x_0)}
\end{aligned}$$

Proof of Theorem 1:

(Case $k=1$)

Choose $h>0$ s.t. $\overline{B_R}(x_0) \subset \Omega_{|h|}$. If $\zeta \in C_c^\infty(\Omega)$ then $\Delta_j^{-h}\zeta \in C_c^\infty\left(\Omega_{|h|}\right)$ and we can use it as our test function in (2):

$$\int_\Omega \sum_{|\alpha|,|\beta|\leq m} (-1)^{|\beta|} a_{\alpha\beta} D^\alpha u D^\beta\left(\Delta_j^{-h}\zeta\right) = \int_\Omega \sum_{|\beta|\leq m} (-1)^{|\beta|} f_\beta D^\beta\left(\Delta_j^{-h}\zeta\right)$$

using fact (c)

$$\int_\Omega \sum_{|\alpha|,|\beta|\leq m} (-1)^{|\beta|} a_{\alpha\beta} D^\alpha u \Delta_j^{-h}\left(D^\beta\zeta\right) = \int_\Omega \sum_{|\beta|\leq m} (-1)^{|\beta|} f_\beta \Delta_j^{-h}\left(D^\beta\zeta\right)$$

using fact (d)

$$-\int_\Omega \sum_{|\alpha|,|\beta|\leq m} (-1)^{|\beta|} \Delta_j^h\left(a_{\alpha\beta}D^\alpha u\right) D^\beta\zeta = -\int_\Omega \sum_{|\beta|\leq m} (-1)^{|\beta|} \Delta_j^h\left(f_\beta\right) D^\beta\zeta$$

using fact (b) and (c)

$$-\int_\Omega \sum_{|\alpha|,|\beta|\leq m} (-1)^{|\beta|} \left(\Delta_j^h\left(a_{\alpha\beta}\right) D^\alpha u + \tilde{a}_{\alpha\beta} D^\alpha\left(\Delta_j^h u\right)\right) D^\beta\zeta = -\int_\Omega \sum_{|\beta|\leq m} (-1)^{|\beta|} \Delta_j^h\left(f_\beta\right) D^\beta\zeta$$

with some variable rebranding:

$$\int_\Omega \sum_{|\alpha|,|\beta|\leq m} (-1)^{|\beta|} \tilde{a}_{\alpha\beta} D^\alpha v_h D^\beta\zeta = \int_\Omega \sum_{|\beta|\leq m} (-1)^{|\beta|} F_{\beta,h} D^\beta\zeta \tag{9}$$

where $\tilde{a}_{\alpha\beta}(x) = a_{\alpha\beta}(x+he_j)$, $v_h = \Delta_j^h u$ and

$$F_{\beta,h} = \Delta_j^h\left(f_\beta\right) - \sum_{|\alpha|\leq m}\left(\Delta_j^h a_{\alpha\beta}\right) D^\alpha u.$$

Ah! Now we can use our helpful lemma (Lemma 1) on (8) [$\tilde{a}_{\alpha\beta}$ still satisfies the ellipticity condition (E) and boundedness (B) because it is just $a_{\alpha\beta}$ at a different point in Ω.] For any $\theta \in (0,1)$

$$\|v_h\|_{m,B_{\theta R}(x_0)} \leq C\left(\|v_h\|_{m-1,B_R(x_0)} + \sum_{|\beta|\leq m}\|F_{\beta,h}\|_{0,B_R(x_0)}\right),$$

where C does not depend on h. Because of (e) and (f) [and remembering that we made some extra assumptions on f_β in the statement of the theorem: $f_\beta \in H^k(\Omega)$] we have that

$$\sup_{B_{R-|h|}(x_0)}\left|\Delta_j^h a_{\alpha\beta}\right| \leq M \quad \text{and} \quad \|\Delta_j^h f_\beta\|_{0,B_{R-|h|}(x_0)} \leq \|f_\beta\|_{1,B_R(x_0)} \tag{10}$$

which we use to rewrite an upper estimate of $\|F_{\beta,h}\|_{0,B_R(x_0)}$ by applying Cauchy's inequality:

$$\|F_{\beta,h}\|_{0,B_R(x_0)} \leq \|f_\beta\|_{1,B_R(x_0)} + KM^2\|u\|_{m,B_R(x_0)}.$$

This implies for some new constant C independent of h

$$\|v_h\|_{m,B_{\theta R}(x_0)} \leq C\left(\|u\|_{m,B_R(x_0)} + \sum_{|\beta|\leq m}\|f_\beta\|_{1,B_R(x_0)}\right).$$

So we get that

$$\limsup_{h\downarrow 0}\|v_h\|_{m,B_{\theta R}(x_0)}$$

exists and is less than some constant which depends on θ, and some other stuff. We now apply:

Lemma 7 (Ch. 5):
If $u \in L^2_{loc}(\Omega)$*, and if*

$$\limsup_{h\downarrow 0} \|\Delta_j^h u\|_{L^2(K)} \le c_K \; (<\infty)$$

for each compact $K \subset \Omega$*. Then* u *has a weak derivative* $D_j u \in L^2_{loc}(\Omega)$*, and* $\Delta_j^h u \rightharpoonup D_j u$ *in the weak sense*

$$\left\langle \Delta_j^h u, \varphi \right\rangle_{L^2} \to \left\langle D_j u, \varphi \right\rangle_{L^2} \quad \forall\, \varphi \in C_c^\infty(\Omega).$$

Further, if $c_K = c$*,* c *independent of* K*, then* $D_j u \in L^2(\Omega)$*.*

Proof:

Take a nested sequence of compact sets $K_i \subset \Omega$ s.t. $K_i \subset K_k$ whenever $i \le k$, that exhaust Ω: $\bigcup_{i>0} K_i = \Omega$. Observe that $L^2(K_i)$ are Banach spaces so that any sequence of L^2-norm bounded functions will have a subsequence that is convergent in $L^2(K_i)$. Since the K_i are nested we can actually get a sequence $\Delta_j^{h_k} u$ which converges in every $L^2(K_i)$. Define $D_j u$ to be the function that the sequence $\Delta_j^{h_k} u$ converges to.

Now we have to show that $\Delta_j^h u$ converges in the weak sense to $D_j u$. Define a function $N : \mathbb{R} \to \mathbb{Z}_+$ by

$$N(h) = \sup \{ h_k \mid k \in \mathbb{Z}_+,\ h_k \le h \}$$

and write

$$\begin{aligned}\left\langle \Delta_j^h u - D_j u, \varphi \right\rangle_{L^2} &= \left\langle \Delta_j^h u - \Delta_j^{N(h)} u + \Delta_j^{N(h)} u - D_j u, \varphi \right\rangle_{L^2} \\ &= \left\langle \Delta_j^h u - \Delta_j^{N(h)} u, \varphi \right\rangle + \left\langle \Delta_j^{N(h)} u - D_j u, \varphi \right\rangle.\end{aligned}$$

The second term clearly goes to 0 b/c $\Delta_j^{N(h)} u \to D_j u$ in the L^2-norm. We want to show that the first term goes to 0 also as $h \downarrow 0$. We write using fact (d):

$$\left\langle \Delta_j^h u - \Delta_j^{N(h)} u, \varphi \right\rangle \;\le\; -\left\langle u, \left(\Delta_j^{-h} - \Delta_j^{-N(h)}\right) \varphi \right\rangle,$$

and the right hand side clearly goes to 0 as $h \downarrow 0$, because φ is differentiable.

Finally, if $c_K = c$ independent of K, then the sequence

$$A_i := \lim_{k\to\infty} \int_{K_i} \left| \Delta_j^{h_k} u \right|^2$$

will be bounded by c, and since K_i exhaust Ω, $\|D_j u\|_{L^2(\Omega)} \le c$.

to show that v_h is weakly convergent to some $v = D_j u$. Similarly by (9) and our Lemma 7, $\Delta_j^h a_{\alpha\beta}$ and $\Delta_j^h f_\beta$ weakly converge to $D_j a_{\alpha\beta}$ and $D_j f_\beta$ resp. in $L^2(B_{\theta R}(x_0))$.

So $u \in H^{m+1}_{\text{loc}}(B_R(x_0))$ and we can pass to the limit in (8) to get that $D_j u$ satisfies

$$\int_\Omega \sum_{|\alpha|,|\beta|\le m} (-1)^{|\beta|}\, \tilde{a}_{\alpha\beta} D^\alpha (D_j u)\, D^\beta \zeta = \int_\Omega \sum_{|\beta|\le m} (-1)^{|\beta|} F_\beta D^\beta \zeta \tag{11}$$

where

$$F_\beta = D_j f_\beta - \sum_{|\alpha|\le m} (D_j a_{\alpha\beta})\, D^\alpha u.$$

Thus summing over $j = 1, \ldots, n$:

$$\|u\|_{m+1, B_{\theta R}(x_0)} \le C \left(\|u\|_{m, B_R(x_0)} + \sum_{|\beta|\le m} \|f_\beta\|_{1, B_R(x_0)} \right)$$

with some new constant C depending on m, n, M, μ, θ, R.

We can repeat this procedure starting from (10) provided condition B_2 holds (F_β contains terms with first derivatives of $a_{\alpha\beta}$). In fact we can repeat it at most k times as long as B_k holds, each time with possibly a different value of θ. Since these θ's are completely arbitrary we can produce

$$\|u\|_{m+k,B_{\theta R}(x_0)} \leq C\left(\|u\|_{m,B_R(x_0)} + \sum_{|\beta|\leq m}\|f_\beta\|_{k,B_R(x_0)}\right).$$

Now replace R with θR in the above inequality:

$$\|u\|_{m+k,B_{\theta^2 R}(x_0)} \leq C\left(\|u\|_{m,B_{\theta R}(x_0)} + \sum_{|\beta|\leq m}\|f_\beta\|_{k,B_{\theta R}(x_0)}\right)$$

and apply Lemma 1 to the term $\|u\|_{m,B_{\theta R}(x_0)}$ to get

$$\|u\|_{m+k,B_{\theta^2 R}(x_0)} \leq C\left(C'\left(\|u\|_{m-1,B_R(x_0)} + \sum_{|\beta|\leq m}\|f_\beta\|_{0,B_R(x_0)}\right) + \sum_{|\beta|\leq m}\|f_\beta\|_{k,B_{\theta R}(x_0)}\right)$$

and after adjusting the constant C and remembering that θ was arbitrary, we get:

$$\|u\|_{m+1,B_{\theta R}(x_0)} \leq C\left(\|u\|_{m-1,B_R(x_0)} + \sum_{|\beta|\leq m}\|f_\beta\|_{k,B_R(x_0)}\right)$$

which is what we wanted to show.