The Yamabe Problem, Spring 2013

University of Maryland, Department of Mathematics course 748F

HW 1 (29 January, 2013):

Please do not submit this.

Reference: Section 1–3 in Chapter 2 of Petersen, Riemannian Geometry, 2nd Edition, 2006, or Aubin's book, or Besse's book.

Familiarize yourself (to the extent you feel comfortable) with the formula for the curvature tensor in local coordinates. Namely, assume the Riemannian metric g is locally (on some ball U, on which the local coordinates $x_1, \ldots, x_n$ are valid) represented by the expression

$$g|_U = \sum_{i,j} g_{ij}(x) dx^i \otimes dx^j,$$

where $g_{ij}(x)$ are smooth and for each $x \in U$ the matrix $[g_{ij}(x)]$ is positive definite. Then the curvature tensor $R = R^l_{ijk} dx^i \otimes dx^j \otimes dx^k \otimes \frac{\partial}{\partial x^l}$ is given by

$$R^l_{ijk} = \partial_i \Gamma^l_{jk} - \partial_j \Gamma^l_{ik} + \Gamma^s_{jk} \Gamma^l_{is} - \Gamma^s ik \Gamma^l_{js},$$

where

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl} (\partial_j g_{il} + \partial_i g_{jl} - \partial_l g_{ij}),$$

where here $[g^{ij}]$ denotes the inverse matrix to $[g_{kl}]$.

Deduce from this a formula for the scalar curvature in local coordinates.

Try to convince yourself of the formula for the change in the scalar curvature under conformal deformation. (You might find it useful to use normal coordinates at a point - these are coordinates such that $g_{ij}(x_0) = \delta_{ij}$, and $\partial_k g_{ij}(x_0) = 0$ (but higher derivatives don't necessarily vanish at x_0!).)

Does this recover formulas you are familiar for the Gauss curvature on a 2-dimensional surface?
(E.g., the hyperbolic metric on a disc is conformal to the flat Euclidean metric on the disc with the conformal factor $1/(1-r^2)^2$.)