The Yamabe Problem, Spring 2013

University of Maryland, Department of Mathematics course 748F

HW2:

Please do not submit this.

1. Familiarize yourself with the (strong) maximum principle.

2. Prove that a conformal class cannot admit two metrics whose scalar curvature have fixed but distinct signs.

3. Prove that the first eigenfunction of the conformal Laplacian is smooth and positive.

4. (i) Find a local coordinate expression for the gradient of a function.
(ii) Derive the local coordinate expression for the Levi-Civita connection ∇ of (M,g) (the Γ_{ij}^k defined in HW1).

5. Show that in local coordinates the Laplacian of (M,g) is given by

$$\Delta_g = \frac{1}{\sqrt{\det g}}\partial_j\big(\sqrt{\det g}g^{ij}\partial_i\big),$$

in particular it is a second order elliptic operator in divergence form.

6. Read Section 3 of the Appendix in Petersen (2nd Ed.), and familiarize yourself with the integration by parts formula

$$-\int f_2\Delta_g f_1 dV_g = \int g(\nabla f_1, \nabla f_2)dV_g.$$

7. Compute the scalar curvature of $(S^m \times S^k, g_{S^m}(R) + g_{S^k}(r))$, where $g_{S^n}(\rho)$ denotes the "round metric" on S^n induced by restricting the Euclidean metric on $\mathbb{R}^{n+1}$ to $\{x \in \mathbb{R}^{n+1} : |x| = \rho\}$. Is this an Einstein space?