Kahler Manifolds HW1

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Problem 1

a) Show octonion multiplication is not associative.

Together with distributivity, octonion multiplication is defined by the following multiplication table:

	i	j	k	l	il	jl	kl
i	-1	k	$-j$	il	$-l$	$-kl$	jl
j	$-k$	-1	i	jl	kl	$-l$	$-il$
k	j	$-i$	-1	kl	$-jl$	il	$-l$
l	$-il$	$-jl$	$-kl$	-1	i	j	k
il	l	$-kl$	jl	$-i$	-1	$-k$	j
jl	kl	l	$-il$	$-j$	k	-1	-1
kl	$-jl$	il	l	$-k$	$-j$	i	-1

Clearly then octonion multiplication is not associative since

$$i\left((il)j\right) = i(-kl) = -jl$$

which is not equal to

$$\left(i(il)\right)j = (-l)j = jl$$

b) Show that, nevertheless, the following weaker form of associativity holds:

$$[a, b, c] := (ab)c - a(bc)$$

vanishes when two of a,b,c are equal. Equivalently, the associator is alternating.

First, we have:

$$\begin{aligned}
[a+b, a+b, c] &= ((a+b)(a+b))c - (a+b)((a+b)c) \\
&= (a^2 + ab + ba + b^2)c - (a+b)(ac+bc) \\
&= (a^2)c + (ab)c + (ba)c + (b^2)c - a(ac) - a(bc) - b(ac) - b(bc) \\
&= [a, a, c] + [a, b, c] + [b, a, c] + [b, b, c]
\end{aligned}$$

Thus by $\mathbb{R}$-linearity and distributivity, we only need to prove $[a, a, c] = [b, b, c] = 0$ and $[a, b, c] + [b, a, c] = 0$ for a, b, c the simple elements i, j, k, l, il, jl, kl. The first, that $[a, a, c] = 0$, is actually assumed in order to define the multiplication table above. The second claim is easy to check directly. This handles the case $[a, a, b] = 0$. The other two cases are similar.

c) Show that octonionic multiplication induces an almost complex structure on the unit imaginary quaternions (hint: use (b)).

In class, the almost complex structure defined on the imaginary unit octonions is given by octonion multiplication $J|_p(v) = pv$ for a point $p \in S^6 = Im\mathbb{O}$ and a tangent vector v. The last thing to check from class is that $J^2 = -1$. By part b), $J^2(v) = p(pv) = (pp)v = -v$ since p is a unit imaginary quaternion.

Problem 2

Hirzebruch signature theorem implies that for a 4-manifold the signature is given by the first Pontryagin class: $\sigma(M) = p_1(M).[M]/3$ (this can also be proved using index theory).
This can be used to show S^4 is not almost complex (reference: D. Aroux, notes to MIT course 966, 2007, lecture 12).
Fill in the details carefully.

The first Pontryagin class of a real vector bundle with a complex structure is defined in terms of the second Chern class of its complexification: $p_1(E) = -c_2(E \otimes \mathbb{C})$. Then we compute:

$$\begin{aligned} p_1(E) &= -c_2(E \otimes \mathbb{C}) \\ &= -c_2(E \oplus \bar{E}) \\ &= -c_1(E)c_1(\bar{E}) - c_2(E) - c_2(\bar{E}) \\ &= c_1(E)^2 - 2c_2(E) \end{aligned}$$

Now assume S^4 has an almost complex structure J. Then from the above, $p_1(TS^4) = c_1(TS42) - 2c_2(TS^4)$. After pairing with the fundamental class $[S^4]$, we get

$$\begin{aligned} c_1(TS^4).[S^4] &= 2c_2(TS^4).[S^4] + 3\sigma(S^4) \\ &= 2\chi(S^4) + 3\sigma(S^4) \\ &= 4. \end{aligned}$$

But $H^2(S^4, \mathbb{Z}) = 0$, so $c_1(TS^4)^2.[S^4] = 0$, a contradiction. We conclude S^4 has no almost complex structure.

Problem 3

Check he claim made in class that the cross product coincides with the standard complex structure on $\hat{\mathbb{C}}$ (the sphere S^2 with the charts $(\mathbb{C}, z)$ and $(\mathbb{C}, w)$ with transition $w = 1/z$ on $\mathbb{C}^*$.

Write $z = x + iy$, and recall that the standard complex structure on $\hat{\mathbb{C}}$ is given by

$$J(\frac{\partial}{\partial x}) = \frac{\partial}{\partial y},$$
$$J(\frac{\partial}{\partial y}) = -\frac{\partial}{\partial x}.$$

In class, we defined an almost complex structure on S^2 embedded in $\mathbb{R}^3$ as follows: for a point $\overrightarrow{x}$ and a tangent vector $\overrightarrow{v}$, $J(\overrightarrow{v}) = \overrightarrow{v} \times \overrightarrow{x}$ (Note that I may have changed the order of the cross-product from class, in order to account for a sign error).

In order to compare the the two, we map $(\mathbb{C}, z)$ onto S^2 minus the north pole via stereographic projection:

$$z = x + iy \mapsto \left(\frac{2x}{x^2+y^2+1}, \frac{2y}{x^2+y^2+1}, \frac{x^2+y^2-1}{x^2+y^2+1} \right).$$

Then the differential maps the tangent vectors $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ into $\mathbb{R}^3$ as

$$\frac{\partial}{\partial x} \mapsto \left(\frac{-2x^2+2y^2+2}{(x^2+y^2+1)^2}, \frac{-4xy}{(x^2+y^2+1)^2}, \frac{4x}{(x^2+y^2+1)^2} \right)$$
$$\frac{\partial}{\partial y} \mapsto \left(\frac{-4xy}{(x^2+y^2+1)^2}, \frac{2x^2-2y^2+2}{(x^2+y^2+1)^2}, \frac{4y}{(x^2+y^2+1)^2} \right)$$

Identifying the point z and the tangent vectors $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$ with their images in $\mathbb{R}^3$ via the above maps, we compute

$$\begin{aligned}
\frac{\partial}{\partial x} \times z &= \left(\frac{-2x^2+2y^2+2}{(x^2+y^2+1)^2}, \frac{-4xy}{(x^2+y^2+1)^2}, \frac{4x}{(x^2+y^2+1)^2} \right) \times \left(\frac{2x}{x^2+y^2+1}, \frac{2y}{x^2+y^2+1}, \frac{x^2+y^2-1}{x^2+y^2+1} \right) \\
&= \frac{1}{(x^2+y^2+1)^3} \left(-4xy(x^2+y^2-1) - 8xy, 8x^2 - (x^2+y^2-1)(-2x^2+2y^2+2), 2(-2x^2+2y^2+2) + 8x^2y \right) \\
&= \frac{1}{(x^2+y^2+1)^3} \left(-4xy(x^2+y^2+1), (2x^2-2y^2+2)(x^2+y^2+1), 4y(x^2+y^2+1) \right) \\
&= \frac{\partial}{\partial y}
\end{aligned}$$

and similarly $\frac{\partial}{\partial y} \times z = -\frac{\partial}{\partial x}$ verifying the claim.

Problem 4

Show that an almost complex manifold is even dimensional.

Let (M, J) be an almost complex manifold. To compute the dimension of M, we compute the dimension of it's tangent space at a point $p \in M$. By definition $J|_P$ (which we will also call J) is an endomorphism of the real vector space T_pM satisfying $J^2 = -1$. Then J has minimal polynomial x^2+1. As such, the eigenvalues of J are $\pm\sqrt{-1}$. Since the coefficients of the characteristic polynomial $p(x) = det(Ix - J)$ are real, its roots $\sqrt{-1}$ and $-\sqrt{-1}$ occur with equal multiplicity, i.e. TpM, and hence M, is even dimensional.

Problem 5

Let (M, ω) be symplectic. Show there exists an almost complex structure J satisfying $\omega(JX, JY) = \omega(X, Y)$ for all X, Y.

First we show that given a symplectic vector space (V, ω), there is a complex structure J on V that is compatible with ω. To do this, first fix a positive definite inner product g. Then g and ω give linear isomorphisms $V \to V^*$ given by:

$$\begin{aligned} V \ni X &\mapsto g(X, \cdot) \in V^* \\ V \ni Y &\mapsto \omega(Y, \cdot) \in V^* \end{aligned}$$

Given $Y \in V$, there is an $X \in V$ such that $g(X, \cdot) = \omega(Y, \cdot)$. The assignment $Y \mapsto X$ is a linear isomorphism $A : V \to V$, i.e. $g(AX, \cdot) = \omega(X, \cdot)$. Also, A is skew-symmetric, since

$$\begin{aligned} g(A^*X, Y) &= g(X, AY) \\ &= g(AY, X) \\ &= \omega(Y, X) \\ &= -\omega(X, Y) \\ &= -g(AX, Y) \end{aligned}$$

A would be a candidate for our complex structure J, except we don't know if it is compatible with ω or satisfies $A^2 = -1$. However, we have the polar decomposition $A = \sqrt{AA^*}J$, i.e. $J = \sqrt{AA^*}^{-1}A$ (Note that AA^* is symmetric and positive definite, so it's square root is defined). I claim J is our complex structure. First, A commutes with AA^*, and hence also $\sqrt{AA^*}^{-1}$. Then

$$\begin{aligned} J^2 &= \sqrt{AA^*}^{-1}A\sqrt{AA^*}^{-1}A \\ &= AA(AA^*)^{-1} \\ &= -AA^*(AA^*)^{-1} \\ &= -1 \end{aligned}$$

And finally J is compatible with ω:

$$\begin{aligned} \omega(JX, JY) &= g(AJX, JY) \\ &= g(JAX, JY) \\ &= g(J * JAX, Y) \\ &= g(AX, Y) \\ &= \omega(X, Y). \end{aligned}$$

Now let (M, ω) be a symplectic manifold. Fix a Riemannian metric g on M. Then locally, the above construction gives a smoothly varying almost complex structure J. The only remaining question is if this J is globally defined. Since the above construction depends only on g and ω, J is canonically, and hence globally, defined.

Problem 6

Let M be a $2n$-dimensional manifold. Show M admits and almost complex structure iff it admits a nondegenerate 2-form (i.e. a form α such that α^n is nowhere zero).

Problem 5 already implies that the existence of a nondegenerate 2-form implies the existence of an almost complex structure, since we never used the condition that ω is closed.

For the reverse direction, fix a Riemannian metric g and define $\omega(X, Y) = g(JX, Y)$. Then $\omega(X, Y) = 0$ for all $Y \in T_pM$ implies $JX = 0$, since g is nondegenerate (in particular, g is positive definite). Then $X = 0$, so ω is nondegenerate.