Spring 2022 Topics in stochastic dynamics

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This is a course on calculus on probability space.

A toy example is $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n), \mu_n)$ with μ_n the standard Gaussian measure on $\mathbb{R}^n$, i.e., for any set $E \subset \mathbb{R}^n$,

$$\mu_n[E] = \int_E \frac{1}{(2\pi)^{\frac{n}{2}}} e^{-\frac{|x|^2}{2}} dx.$$

This is a classical object that appears in the central limit theorem. In the course we will be mostly interested in the infinite dimensional case, i.e., when the Gaussian measure is defined on some infinite dimensional function space. A canonical example throughout the course is the following: consider a Gaussian process $V = \{V(x)\}_{x\in\mathbb{R}^d}$ (a family of jointly Gaussian random variables indexed by $x \in \mathbb{R}^d$) and a sequence of random variables constructed from V which we write as $X_n = f_n(V)$ for some complicated *deterministic* function f_n that can be both nonlinear and nonlocal. The question is to understand how the randomness from V propagates into X_n through the mapping f_n, in particular, what is the mean, variance, and asymptotic distribution of X_n as $n \to \infty$. The simplest example is when $V = \{V_1, V_2, \ldots\}$ with V_i i.i.d. standard normal random variables, and $X_n = V_1 + \ldots + V_n$, which is naturally connected to law of large numbers and central limit theorems.

Given the Gaussian process V, the answer to the question should only depend on the properties of f_n, which is a sequence of *deterministic* functions, so the problem reduces to understand what properties of f_n matter. Like in calculus when studying properties of a smooth function, a first object we define here is some type of derivative of f_n with respect to V, then we proceed to build a calculus out of it. A recurring theme is to understand how the randomness behaves under complicated nonlinear transformations given by dynamical system, PDE or models from statistical mechanics. A key tool will be an integration by parts formula in the infinite dimensional space. The applications covered in the course will include (i) a general criterion of proving central limit theorem which is the so-called "2nd order Poincaré inequality", (ii) analyzing partial differential equation subject to random perturbations (think of V as the perturbation and X_n is the solution to the PDE), (iii) integration by parts in infinite dimensional space and applications to models in statistical mechanics. We will start with the discrete setting, where the underlying Gaussian space is given by a sequence of i.i.d. $N(0, 1)$ random variables so everything is more explicit. Then we will move to the continuous setting.

(1) **Time**: TuTh 3:30pm - 4:45pm

(2) **Place**: PHY 2124

(3) **Office hours**: TuTh 2:30pm - 3:30pm/after the lecture/email for appointments/discuss on Piazza.

(4) **Prerequisite**: basic probability and real analysis

(5) **References**:
- Chapter 1-5 of the notes at http://www.math.utah.edu/~davar/math7880/F18/GaussianAnalysis.pdf
- The notes at http://www.hairer.org/notes/Malliavin.pdf
- *David Nualart*, "Malliavin Calculus and Its Applications". *Electronic version can be found on Springer through UMD library*

(6) **Tentative schedule**:
  (a) Gaussian measure and white noise
  (b) Derivative and adjoint operator
  (c) Hermite polynomial expansion and Wiener chaos
  (d) Heat flow and Ornstein-Uhlenbeck operator
  (e) Integration by parts
  (f) Stein's method and second order Poincaré inequality
  (g) Continuous Gaussian process
  (h) Applications to random PDE

(7) **Homeworks**:
  - There are 5 or 6 homeworks, each containing 5 or 6 problems. Assigned on Canvas, approximately every two weeks.
  - You are encouraged to work in groups. I encourage questions/discussions on Piazza. Homeworks are collected on Canvas. Late homeworks are NOT accepted

(8) **Presentation**: I will assign multiple papers to read throughout the semester, and you can choose the interested topics, and prepare a presentation based on the papers you read.

(9) **Grades**: There are two ways of calculating the grade you can choose: (i) 100% homeworks (ii) 50% homeworks and 50% presentation (in this case you only need to do half of the homework problems)