FLUCTUATIONS OF A NONLINEAR STOCHASTIC HEAT EQUATION IN DIMENSIONS THREE AND HIGHER

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ABSTRACT. We study the solution to a nonlinear stochastic heat equation in $d \geq 3$. The equation is driven by a Gaussian multiplicative noise that is white in time and smooth in space. For a small coupling constant, we prove (i) the solution converges to the stationary distribution in large time; (ii) the diffusive scale fluctuations are described by the Edwards-Wilkinson equation.

KEYWORDS: Stochastic heat equation, Malliavin calculus, stationary solution.

1. INTRODUCTION

1.1. **Main result.** We study the solution to the nonlinear stochastic heat equation

$$\partial_t u = \Delta u + \beta\sigma(u)\dot{W}_\phi(t,x), \qquad t > 0, x \in \mathbb{R}^d, d \geq 3, \tag{1.1}$$

with constant initial data $u(0,x) \equiv 1$, where $\beta > 0$ is a constant. We assume that $\sigma(\cdot)$ is a global Lipschitz function satisfying $|\sigma(x)-\sigma(y)| \leq \sigma_{\mathrm{Lip}}|x-y|$ for all $x, y \in \mathbb{R}^d$. Here σ_{Lip} is a fixed positive constant. Moreover, $\dot{W}_\phi$ is a centered Gaussian noise that is white in time and smooth in space, constructed from a spacetime white noise $\dot{W}$ and a non-negative mollifier $\phi \in C_c^\infty(\mathbb{R}^d)$:

$$\dot{W}_\phi(t,x) = \int_{\mathbb{R}^d} \phi(x-y)\dot{W}(t,y)\mathrm{d}y.$$

The covariance function is given by

$$\mathbb{E}[\dot{W}_\phi(t,x)\dot{W}_\phi(s,y)] = \delta_0(t-s)R(x-y),$$
$$R(x) = \int_{\mathbb{R}^d} \phi(x+y)\phi(y)\mathrm{d}y \in C_c^\infty(\mathbb{R}^d).$$

Under our assumptions, there exists a unique continuous random field as the mild solution to (1.1), given by

$$u(t,x) = 1 + \beta\int_0^t \int_{\mathbb{R}^d} p(t-s,x-y)\sigma(u(s,y))\dot{W}_\phi(s,y)\mathrm{d}y\mathrm{d}s, \tag{1.2}$$

where $p(t,x) = (4\pi t)^{-d/2}e^{-\frac{|x|^2}{4t}}$ is the heat kernel, and the stochastic integral in (1.2) is interpreted in the Itô-Walsh sense. We rescale the solution diffusively, and define

$$u_\varepsilon(t,x) = u\left(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}\right).$$

The first result is about the behavior of u as $t \to \infty$.

Theorem 1.1. *There exists $\beta_0 = \beta_0(d,\phi,\sigma) > 0$ such that if $\beta < \beta_0$, then $u(t,\cdot) \Rightarrow Z(\cdot)$ in $C(\mathbb{R}^d)$, as $t \to \infty$, where $Z(\cdot)$ is a stationary random field.*

On top of this result, we obtain the Edwards-Wilkinson limit as follows:

Theorem 1.2. *Under the same assumption of Theorem 1.1, for any test function $g \in C_c^\infty(\mathbb{R}^d)$ and $t > 0$, we have*

$$\frac{1}{\varepsilon^{\frac{d}{2}-1}} \int_{\mathbb{R}^d} (u_\varepsilon(t,x) - 1)\, g(x)\mathrm{d}x \Rightarrow \int_{\mathbb{R}^d} \mathcal{U}(t,x) g(x)\mathrm{d}x$$

in distribution as $\varepsilon \to 0$, where $\mathcal{U}$ solves the Edwards-Wilkinson equation

$$\partial_t \mathcal{U} = \Delta \mathcal{U} + \beta \nu_\sigma \dot{W}(t,x), \qquad \mathcal{U}(0,x) \equiv 0,$$

and ν_σ is the effective constant depending on σ, the spatial covariance function R, and the stationary random field Z obtained in Theorem 1.1:

$$\nu_\sigma^2 = \int_{\mathbb{R}^d} \mathbb{E}[\sigma(Z(0))\sigma(Z(x))] R(x)\mathrm{d}x. \tag{1.3}$$

1.2. **Context.** The linear version of (1.1) was studied in [9, 14, 19, 21]: for small β and the equation

$$\partial_t u = \Delta u + \beta u \dot{W}_\phi(t,x), \tag{1.4}$$

results similar to Theorem 1.1 and 1.2 were proved: (i) the pointwise distribution of $u(t,x)$ converges as $t \to \infty$; (ii) as a random Schwartz distribution, $\varepsilon^{1-\frac{d}{2}}[u_\varepsilon(t,\cdot) - 1]$ converges to the Gaussian field given by the solution to the Edwards-Wilkinson equation. Through a Hopf-Cole transformation $h = \log u$, a KPZ-type of equation

$$\partial_t h = \Delta h + |\nabla h|^2 + \beta \dot{W}_\phi(t,x) \tag{1.5}$$

was also studied, and the same Edwards-Wilkinson limit was established in [9, 11, 19, 20], see also [8]. Similar results were proved in [2, 3, 4, 13] when $d = 2$, where the coupling constant β is tuned logarithmically in ε. The previous studies of the nonlinear equation (1.5) all rely on the Hopf-Cole transformation and the fact that the solution to the linear equation (1.4) can be written explicitly by the Feynman-Kac formula or the Wiener chaos expansion. In light of the Hairer-Quastel universality result in the subcritical setting [15], it is very natural to ask that, in the present critical setting, if we can study a more general Hamilton-Jacobi equation

$$\partial_t h = \Delta h + H(\nabla h) + \beta \dot{W}_\phi(t,x), \tag{1.6}$$

where the Hamiltonian H is not necessarily quadratic, and prove a similar result of convergence to the Edwards-Wilkinson equation, for small β. The only result in this direction that we are aware of is a two-dimensional anisotropic KPZ equation studied in [1], where the authors considered the nonlinearity $H(\nabla h) = (\partial_{x_1} h)^2 - (\partial_{x_2} h)^2$ and proved the existence of subsequential limits of the solutions started from an invariant measure.

In this short note, we study (1.1), which to some extent sits between the linear equation (1.4) and the nonlinear equation (1.6). The nonlinear term $\sigma(u)$ excludes the use of the Feynman-Kac formula or the Wiener chaos expansion as in the case of (1.4), so the previous approaches do not apply. Meanwhile, (1.1) is less nonlinear compared to (1.6), and we are able to make a substantial use of the mild formulation (1.2).

Part of our approach is inspired by another line of work, where similar results were proved for the spatial averages of $u(t,\cdot)$ [16, 17, 23]. For a large class of equations and noises, which in particular covers (1.1), central limit theorems were proved for the random variables

$$\varepsilon^{-\frac{d}{2}} \int_{\mathbb{R}^d} [u(t, \tfrac{x}{\varepsilon}) - 1] g(x)\mathrm{d}x.$$

Studying the scaling $(t,x) \mapsto (\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon})$ as in our case requires a good understanding of the local statistics of $u(t,x)$ as $t \to \infty$, and this is provided by Theorem 1.1 by proving the convergence to a stationary distribution. The local statistical property of $Z(\cdot)$ appears naturally in the expression of the effective variance (1.3), see the heuristic argument at the beginning of Section 4.

For the linear stochastic heat equation (1.4) in $d \geq 3$ with $\beta \ll 1$, the convergence to the stationary solution was shown in [12, 18], based on the Feynman-Kac formula. For semilinear equations, the existence of stationary solutions/invariant measures was proved e.g. in the early work [10, 24], but the convergence to the invariant measure as stated in Theorem 1.1 seems to be unknown. Although our main focus of the paper is on the constant initial data, a similar proof works for more general cases. In Remark 3.6 below, we explain how to adapt the proof of Theorem 1.1 to cover the example of "small" perturbations of the constant initial data.

It is worth mentioning that the assumption of small β is necessary for the result to hold. We know from [21] that the pointwise distribution of $u(t,x)$ converges to zero as $t \to \infty$, if β is beyond a critical value. The recent works [9, 19] extend the result in Theorem 1.2 to the whole regime of β in which $\nu_\sigma < \infty$, in the linear case of $\sigma(x) \equiv x$.

Organization of the paper. In Section 2, we introduce the basic tools of analysis on Gaussian space and prove some estimates on the solution u as well as its Malliavin derivative that are used in the sequel. The proofs of Theorems 1.1 and 1.2 are in Sections 3 and 4 respectively.

Notations. We use the following notations and conventions throughout this paper.

(i) We use $a \lesssim b$ to denote $a \leq Cb$ for some constant C that is independent of ε. For instance, as σ is global Lipschitz, we have $|\sigma(x)| \lesssim 1 + |x|$.

(iii) $\|\cdot\|_p$ denotes the $L^p(\Omega)$ norm of the probability space $(\Omega, \mathscr{G}, \mathbb{P})$ where the spacetime white noise $\dot{W}$ is built on.

(iv) $p(t,x) = (4\pi t)^{-d/2} e^{-\frac{|x|^2}{4t}}$ is the heat kernel of $\partial_t - \Delta$.

(v) The Fourier transform of f is denoted by $\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-i\xi\cdot x} \mathrm{d}x$.

Acknowledgement. We thank the two anonymous referees for many helpful suggestions to improve the presentation. The work was partially supported by the NSF through DMS-1907928 and the Center for Nonlinear Analysis of CMU.

2. Preliminaries

Throughout this paper, we consider the centered Gaussian noise $\dot{W}_\phi(t,x)$ on $\mathbb{R} \times \mathbb{R}^d$ with $d \geq 3$, whose covariance is given by

$$\mathbb{E}[\dot{W}_\phi(t,x)\dot{W}_\phi(s,y)] = \delta_0(t-s)R(x-y),$$

where the spatial covariance function R is assumed to be smooth and has a compact support. One may associate an isonormal Gaussian process to this noise. Consider a stochastic process

$$\left\{W_\phi(h) = \int_{\mathbb{R}^{1+d}} h(s,x)\dot{W}_\phi(s,x)\mathrm{d}x\mathrm{d}s, \quad h \in C_c^\infty(\mathbb{R} \times \mathbb{R}^d)\right\}$$

defined on a complete probability space $(\Omega, \mathscr{G}, \mathbb{P})$ satisfying

$$\mathbb{E}[W_\phi(h)W_\phi(g)] = \int_{-\infty}^{\infty}\int_{\mathbb{R}^{2d}} h(s,x)g(s,y)R(x-y)\mathrm{d}x\mathrm{d}y\mathrm{d}s.$$

As R is positive definite, the above integral defines an inner product, which we denote by $\langle \cdot, \cdot \rangle_{\mathcal{H}}$, so that $\mathbb{E}[W_\phi(h)W_\phi(g)] = \langle h, g\rangle_{\mathcal{H}}$ for all $h, g \in C_c^\infty(\mathbb{R} \times \mathbb{R}^d)$. Complete the space $C_c^\infty(\mathbb{R}\times\mathbb{R}^d)$ with respect to this inner product, and denote the completion by $\mathcal{H}$, and thus we obtain an isonormal Gaussian process $\{W_\phi(h), h\in\mathcal{H}\}$. Consider the σ-algebra defined by

$$\mathscr{F}_t^0 := \sigma\{W_\phi(\mathbb{1}_{[0,s]}(\cdot)\mathbb{1}_A(\cdot)) : 0\le s\le t, A\in\mathscr{B}_b(\mathbb{R}^d)\},$$

where $\mathscr{B}_b(\mathbb{R}^d)$ denotes the bounded Borel subsets of $\mathbb{R}^d$ and let $\mathscr{F}_t$ denote the completion of $\mathscr{F}_t^0$ with respect to the measure $\mathbb{P}$. Denote $\mathscr{F} = \{\mathscr{F}_t : t\ge 0\}$, which is the natural filtration generated by $\dot{W}_\phi$, and then for all $\mathscr{F}$-adapted, jointly measurable random field $\{\Phi(t,x) : (t,x)\in\mathbb{R}\times\mathbb{R}^d\}$ such that

$$\mathbb{E}[\|\Phi\|_{\mathcal{H}}^2] = \mathbb{E}\left[\int_{-\infty}^{\infty}\int_{\mathbb{R}^{2d}}\Phi(t,x)\Phi(t,y)R(x-y)\mathrm{d}x\mathrm{d}y\mathrm{d}t\right] < \infty,$$

the stochastic integral

$$\int_{-\infty}^{\infty}\int_{\mathbb{R}^d}\Phi(t,x)\mathrm{d}W_\phi(t,x)$$

is well-defined in the Itô-Walsh sense, and the Itô isometry holds:

$$\mathbb{E}\left[\left|\int_{-\infty}^{\infty}\int_{\mathbb{R}^d}\Phi(t,x)\mathrm{d}W_\phi(t,x)\right|^2\right] = \mathbb{E}[\|\Phi\|_{\mathcal{H}}^2]. \tag{2.1}$$

Throughout the paper, we will not distinguish the following two expressions:

$$\int\Phi(t,x)\dot{W}_\phi(t,x)\mathrm{d}x\mathrm{d}t \quad\text{and}\quad \int\Phi(t,x)\mathrm{d}W_\phi(t,x).$$

In the proof of Theorem 1.2, we also need to adopt methods from Malliavin calculus, so let us introduce a differential structure on the infinite-dimensional space in the manner of Malliavin. We shall follow the notations from [22]. Let $\mathcal{S}$ be the space of random variables of the form $F = f(W_\phi(h_1),\cdots,W_\phi(h_n))$, where $f\in C^\infty(\mathbb{R}^n)$ with all derivatives having at most polynomial growth. Its Malliavin derivative is an $\mathcal{H}$-valued random variable given by

$$DF = \sum_{i=1}^n \partial_i f(W_\phi(h_1),\cdots,W_\phi(h_n))h_i,$$

where $\partial_i f$ denotes the partial derivative of f with respect to the i-th variable. By induction, one may define the higher-order derivative $D^l F$, $l = 1,2,\cdots$, which is an $\mathcal{H}^{\otimes l}$-valued random variable. Then the Sobolev norm $\|\cdot\|_{r,p}$ of such an F is defined as

$$\|F\|_{r,p} = \left(\mathbb{E}[|F|^p] + \sum_{l=1}^r \mathbb{E}[\|D^l F\|_{\mathcal{H}^{\otimes l}}^p]\right)^{\frac{1}{p}}.$$

Complete $\mathcal{S}$ with respect to this Sobolev norm and denote the completion by $\mathbb{D}^{r,p}$.

Let δ be the divergence operator, which is the adjoint operator of the differential operator D. For each $v\in\mathrm{Dom}\delta$, define $\delta(v)$ to be the unique element in $L^2(\Omega)$ such that

$$\mathbb{E}[F\delta(v)] = \mathbb{E}[\langle DF, v\rangle_{\mathcal{H}}], \quad \forall F\in\mathbb{D}^{1,2}. \tag{2.2}$$

For $v\in\mathrm{Dom}\delta$, $\delta(v)$ is also called the Skorokhod integral of v. In our case of v being adapted to the filtration $\mathcal{F}_t$, it coincides with the Walsh integral, which is written as $\delta(v) = \int v(t,x)\mathrm{d}W_\phi(t,x)$. The Malliavin derivative $D_{t,x}\delta(v) = D\delta(v)(t,x)$ is given by (see Proposition 1.3.8, Chapter 1, [22])

$$D_{t,x}\delta(v) = v(t,x) + \int_{-\infty}^{\infty}\int_{\mathbb{R}^d} D_{t,x}v(s,y)\mathrm{d}W_\phi(s,y).$$

Using the mild formulation (1.2) of the nonlinear stochastic heat equation, we may write

$$u(t,x) = 1 + \delta(v_{t,x}),$$

where

$$v_{t,x}(s,y) = \beta 1\!\!1_{[0,t]}(s) p(t-s, x-y)\sigma(u(s,y)),$$

so the Malliavin derivative of the solution u is given by

$$\begin{aligned} D_{r,z}u(t,x) =& \beta 1\!\!1_{[0,t]}(r) p(t-r, x-z)\sigma(u(r,z)) \\ &+ \beta \int_r^t \int_{\mathbb{R}^d} p(t-s, x-y)\Sigma(s,y) D_{r,z}u(s,y) \mathrm{d}W_\phi(s,y). \end{aligned}$$

where $\Sigma(s,y)$ is an adapted process, bounded by the Lipschitz constant σ_{Lip}. If we further assume that $\sigma(\cdot)$ is continuously differentiable, then $\Sigma(s,y) = \sigma'(u(s,y))$.

We first prove the following moment estimates on u. This result is very important as it will be applied in the proofs of both Theorem 1.1 and Theorem 1.2.

Lemma 2.1. *For any $p > 1$, there exists $\beta_0 = \beta_0(d, p, \phi, \sigma_{\mathrm{Lip}})$ and $C = C(\beta_0, d, p, \phi, \sigma_{\mathrm{Lip}})$ such that if $\beta < \beta_0$, we have*

$$\sup_{t \geq 0} \|u(t,0)\|_p \leq C.$$

Proof. By the mild formulation,

$$u(t,0) = 1 + \beta \int_0^t \int_{\mathbb{R}^d} p(t-s, y)\sigma(u(s,y)) \mathrm{d}W_\phi(s,y).$$

For any $n \in \mathbb{Z}_{\geq 0}$, by the Burkholder-Davis-Gundy inequality and the Minkowski inequality, we have

$$\begin{aligned} \|u(t,0)\|_{2n}^{2n} \lesssim& 1 + \beta^{2n} \left\| \int_0^t \int_{\mathbb{R}^{2d}} p(t-s,y_1)p(t-s,y_2)\sigma(u(s,y_1))\sigma(u(s,y_2))R(y_1-y_2)\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}s \right\|_n^n \\ \leq& 1 + \beta^{2n} \left(\int_0^t \int_{\mathbb{R}^{2d}} p(t-s,y_1)p(t-s,y_2)\|\sigma(u(s,y_1))\sigma(u(s,y_2))\|_n R(y_1-y_2)\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}s \right)^n. \end{aligned}$$

Applying Hölder's inequality and using the stationarity of $u(s,\cdot)$, we further obtain

$$\|\sigma(u(s,y_1))\sigma(u(s,y_2))\|_n \leq \|\sigma(u(s,0))\|_{2n}^2 \lesssim 1 + \|u(s,0)\|_{2n}^2.$$

Thus, if we define $f(t) = \|u(t,0)\|_{2n}^2$, then

$$f(t) \lesssim 1 + \beta^2 \int_0^t \int_{\mathbb{R}^{2d}} p(t-s,y_1)p(t-s,y_2)(1+f(s))R(y_1-y_2)\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}s.$$

For the integration in y_1, y_2, we use the elementary inequality

$$\int_{\mathbb{R}^{2d}} p(t-s,y_1)p(t-s,y_2)R(y_1-y_2)\mathrm{d}y_1\mathrm{d}y_2 \lesssim 1 \wedge (t-s)^{-d/2},$$

which yields the integral inequality for f: there exists $C > 0$ independent of t such that

$$f(t) \leq C + C\beta^2 \int_0^t [1 \wedge (t-s)^{-d/2}] f(s)\mathrm{d}s. \tag{2.3}$$

As the kernel $1 \wedge s^{-d/2}$ is in $L^1(\mathbb{R}_+)$ in $d \geq 3$, we choose β small so that

$$C\beta^2 \int_0^\infty [1 \wedge s^{-d/2}]\mathrm{d}s < 1$$

and a direct iteration of (2.3) shows that $\sup_{t \geq 0} f(t) \lesssim 1$, which completes the proof. □

Next, we establish an upper bound of $\|D_{r,z}u(t,x)\|_p$ which will be useful in the proof of Theorem 1.2.

Lemma 2.2. *For all $t > 0$, $x \in \mathbb{R}^d$ and $p \geq 2$, there exists some constant $C = C(\beta, d, p, \phi, \sigma_{\text{Lip}})$ such that*

$$\|D_{r,z}u(t,x)\|_p \leq Cp(t-r, x-z), \quad \text{for all } (r,z) \in (0,t) \times \mathbb{R}^d.$$

Proof. The proof follows from [6, Lemma 3.11], with slight modifications. Let $S_{r,z}(t,x)$ be the solution to

$$S_{r,z}(t,x) = \beta p(t-r, x-z) + \beta \int_r^t \int_{\mathbb{R}^d} p(t-s, x-y)\Sigma(s,y)S_{r,z}(s,y)\mathrm{d}W_\phi(s,y),$$

then due to the uniqueness of the solution to the above equation,

$$D_{r,z}u(t,x) = S_{r,z}(t,x)\sigma(u(r,z)).$$

By the Burkholder-Davis-Gundy inequality,

$$\begin{aligned}\|S_{r,z}(t,x)\|_p^2 \lesssim & \beta^2 p(t-r,x-z)^2 + \beta^2 \int_r^t \int_{\mathbb{R}^{2d}} p(t-s,x-y_1)p(t-s,x-y_2) \\ & \cdot \|S_{r,z}(s,y_1)\|_p \|S_{r,z}(s,y_2)\|_p R(y_1-y_2)\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}s.\end{aligned}$$

Notice that if we set $t = \theta + r$ and $x = \eta + z$, then the above estimate for $S_{r,z}(\theta+r, \eta+z)$ is independent of (r,z), so for convenience we may denote

$$g(\theta,\eta) = \|S_{r,z}(\theta+r, \eta+z)\|_p,$$

and thus

$$\begin{aligned}g(\theta,\eta)^2 \leq & C\beta^2 p(\theta,\eta)^2 + C\beta^2 \int_0^\theta \int_{\mathbb{R}^{2d}} p(\theta-s,\eta-y_1)p(\theta-s,\eta-y_2) \\ & \cdot g(s,y_1)g(s,y_2)R(y_1-y_2)\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}s.\end{aligned}$$

Then according to Lemma 2.7 in [6], we may conclude that

$$g(\theta,\eta) \leq \sqrt{C\beta^2}p(\theta,\eta)H(\theta, 2C\beta^2)^{\frac{1}{2}},$$

and H is defined as

$$H(t,\lambda) = \sum_{n=0}^\infty \lambda^n h_n(t),$$

where $h_0(t) = 1$ and

$$h_n(t) = \int_0^t h_{n-1}(s) \int_{\mathbb{R}^d} p(t-s,z)R(z)\mathrm{d}z\mathrm{d}s, \quad n \geq 1.$$

We notice that for all t,

$$|h_1(t)| \leq \int_0^\infty \int_{\mathbb{R}^d} p(s,z)R(z)\mathrm{d}z\mathrm{d}s =: C_R,$$

and thus for all $n \geq 1$ and $t \geq 0$,

$$|h_n(t)| \leq C_R^n.$$

Therefore, for sufficiently small β we have $H(\theta, 2C\beta^2) \lesssim 1$ uniformly in θ, and hence, it follows that for all $p > 1$,

$$\|D_{r,z}u(t,x)\|_p \leq \|S_{r,z}(t,x)\|_{2p}\|\sigma(u(r,z))\|_{2p} \leq Cp(t-r,x-z),$$

where in the last step we used the fact that $|\sigma(x)| \lesssim 1 + |x|$ and Lemma 2.1. □

In the proof of Theorem 1.2, we need the following result which is also used in [16]. With the help of the this result, to establish the required convergence in law, we only need to control the L^2-distance on the right-hand side of the result below, which is more amenable to calculations.

Proposition 2.3. *Let X be a random variable such that $X = \delta(v)$ for $v \in \operatorname{Dom}\delta$. Assume $X \in \mathbb{D}^{1,2}$. Let Z be a centered Gaussian random variable with variance Σ. For any C^2-function $h : \mathbb{R} \to \mathbb{R}$ with bounded second order derivative, then*

$$|\mathbb{E}h(X) - \mathbb{E}h(Z)| \leq \frac{1}{2}\|h''\|_\infty \sqrt{\mathbb{E}\left[(\Sigma - \langle DX, v\rangle_{\mathcal{H}})^2\right]}.$$

We also need to apply the following version of Clark-Ocone formula in the proof of Theorem 1.2, to estimate certain covariance.

Proposition 2.4 (Clark-Ocone Formula)**.** *Let $X \in \mathbb{D}^{1,2}$, then*

$$X = \mathbb{E}[X] + \int_{\mathbb{R}_+ \times \mathbb{R}^d} \mathbb{E}[D_{r,z}X|\mathscr{F}_r] dW_\phi(r,z).$$

The proof of this formula can be found e.g. in [7, Proposition 6.3].

3. Proof of Theorem 1.1

To prove the convergence of $u(t,\cdot)$ to the stationary distribution, we use a rather standard approach: instead of sending $t \to \infty$ and showing $u(t,\cdot)$ converges *weakly*, we initiate the equation at $t = -K$ and consider the solution at $t = 0$, then send $K \to \infty$ to prove the *strong convergence*.

More precisely, for $K > 0$, we consider a family of equations indexed by K:

$$\begin{aligned} &\partial_t u_K = \Delta u_K + \beta\sigma(u_K)\dot{W}_\phi(t,x), \qquad t > -K, x \in \mathbb{R}^d, \\ &u_K(-K,x) \equiv 1. \end{aligned} \tag{3.1}$$

By the stationarity of the noise $\dot{W}_\phi$, we know that for all $K > 0$, $u(K,\cdot)$ and $u_K(0,\cdot)$ as random variables taking values in $C(\mathbb{R}^d)$, have the same law. Then the problem reduces to proving the weak convergence of $C(\mathbb{R}^d)$-valued random variables $u_K(0,\cdot)$.

The following two propositions combine to complete the proof of Theorem 1.1.

Proposition 3.1. *For each $x \in \mathbb{R}^d$, $\{u_K(0,x)\}_{K\geq 0}$ is a Cauchy sequence in $L^2(\Omega)$.*

Proposition 3.2. *The sequence of $C(\mathbb{R}^d)$-valued random variables $\{u(t,\cdot)\}_{t\geq 0}$ is tight.*

Since $u_K(0,x)$ is a stationary process in x, to show $\{u_K(0,x)\}_{K\geq 0}$ is Cauchy in $L^2(\Omega)$, we only need to consider $x = 0$. We write (3.1) in the mild formulation:

$$u_K(t,x) = 1 + \beta \int_{-K}^{t} \int_{\mathbb{R}^d} p(t-s,x-y)\sigma(u_K(s,y))\mathrm{d}W_\phi(s,y), \quad t > -K, x \in \mathbb{R}^d. \tag{3.2}$$

Therefore, for any $K_1 > K_2 \geq 0$, and $t \geq -K_2$, we can write the difference as

$$u_{K_1}(t,0) - u_{K_2}(t,0) = \beta[I_{K_1,K_2}(t) + J_{K_1,K_2}(t)], \tag{3.3}$$

with

$$I_{K_1,K_2}(t) = \int_{-K_1}^{-K_2} \int_{\mathbb{R}^d} p(t-s,y)\sigma(u_{K_1}(s,y))\mathrm{d}W_\phi(s,y),$$

$$J_{K_1,K_2}(t) = \int_{-K_2}^{t} \int_{\mathbb{R}^d} p(t-s,y)[\sigma(u_{K_1}(s,y)) - \sigma(u_{K_2}(s,y))]\mathrm{d}W_\phi(s,y).$$

For $t > -K_2 > -K_1$, define

$$\alpha_{K_1,K_2}(t) = \int_{t+K_2}^{t+K_1} (1 \wedge s^{-d/2})\mathrm{d}s. \tag{3.4}$$

and

$$\gamma_{K_1,K_2}(t) = \mathbb{E}[|u_{K_1}(t,0) - u_{K_2}(t,0)|^2]. \tag{3.5}$$

We have the following lemmas.

Lemma 3.3. *For any* $t > -K_2$, $\mathbb{E}[|I_{K_1,K_2}(t)|^2] \lesssim \alpha_{K_1,K_2}(t)$.

Lemma 3.4. $\mathbb{E}[|J_{K_1,K_2}(t)|^2] \lesssim \int_{-K_2}^t [1 \wedge (t-s)^{-d/2}]\gamma_{K_1,K_2}(s)\mathrm{d}s$

Proof of Lemma 3.3. By Itô's isometry and the fact that $|\sigma(x)| \lesssim 1 + |x|$, we have

$$\mathbb{E}[|I_{K_1,K_2}(t)|^2] \lesssim \int_{-K_1}^{-K_2} \int_{\mathbb{R}^{2d}} p(t-s,y_1)p(t-s,y_2) \cdot \mathbb{E}[(1+u_{K_1}(s,y_1))(1+u_{K_1}(s,y_2))]R(y_1-y_2)\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}s.$$

Further applying Lemma 2.1 yields

$$\begin{aligned}\mathbb{E}[|I_{K_1,K_2}(t)|^2] &\lesssim \int_{-K_1}^{-K_2} \int_{\mathbb{R}^{2d}} p(t-s,y_1)p(t-s,y_2)R(y_1-y_2)\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}s \\ &\lesssim \int_{-K_1}^{-K_2} [1 \wedge (t-s)^{-d/2}]\mathrm{d}s = \alpha_{K_1,K_2}(t).\end{aligned}$$

□

Proof of Lemma 3.4. By Itô's isometry, the Lipchitz property of σ, and the stationarity of $u_K(s,\cdot)$, we have

$$\mathbb{E}[|J_{K_1,K_2}(t)|^2] \lesssim \int_{-K_2}^{t} \int_{\mathbb{R}^d} p(t-s,y_1)p(t-s,y_2) \cdot \mathbb{E}[|u_{K_1}(s,0) - u_{K_2}(s,0)|^2]R(y_1-y_2)\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}s.$$

After an integration in y_1, y_2, the right-hand side of the above inequality can be bounded by

$$\int_{-K_2}^{t} [1 \wedge (t-s)^{-d/2}]\mathbb{E}[|u_{K_1}(s,0) - u_{K_2}(s,0)|^2]\mathrm{d}s,$$

which completes the proof. □

Combining the above two lemmas with (3.3), we have the integral inequality

$$\begin{aligned}\gamma_{K_1,K_2}(t) =&\beta^2\mathbb{E}[|I_{K_1,K_2}(t) + J_{K_1,K_2}(t)|^2] \\ \le& C\beta^2\alpha_{K_1,K_2}(t) + C\beta^2 \int_{-K_2}^{t} [1 \wedge (t-s)^{-d/2}]\gamma_{K_1,K_2}(s)\mathrm{d}s, \quad \text{for all } t > -K_2,\end{aligned} \tag{3.6}$$

where $C > 0$ is a constant independent of t, K_1, K_2. The following lemma completes the proof of Proposition 3.1.

Lemma 3.5. *For fixed* $t > -K_2$, $\gamma_{K_1,K_2}(t) \to 0$ *as* $K_2 \to \infty$.

Proof. To ease the notation in the proof, we simply write (3.6) as

$$\gamma(t) \le C\beta^2\alpha(t) + C\beta^2 \int_{-K_2}^{t} k(t-s)\gamma(s)\mathrm{d}s, \tag{3.7}$$

where we omitted the dependence on K_1, K_2 and denote $k(s) = 1 \wedge s^{-d/2}$.

Before going to the iteration, we claim the following inequality holds:

$$\int_{-K_2}^{t} k(t-s)[1 \wedge (s+K_2)^{-(\frac{d}{2}-1)}]\mathrm{d}s \lesssim 1 \wedge (t+K_2)^{-(\frac{d}{2}-1)}, \quad \text{for } t \ge -K_2. \tag{3.8}$$

By a change of variable, we have

$$\int_{-K_2}^{t} k(t-s)[1 \wedge (s+K_2)^{-(\frac{d}{2}-1)}]\mathrm{d}s$$
$$= \int_0^{t+K_2} [1 \wedge (t+K_2-s)^{-\frac{d}{2}}][1 \wedge s^{-(\frac{d}{2}-1)}]\mathrm{d}s.$$

First note that the integral is bounded uniformly in $t+K_2$, then we decompose the integration domain:

$$\int_0^{\frac{t+K_2}{2}} [1 \wedge (t+K_2-s)^{-\frac{d}{2}}][1 \wedge s^{-(\frac{d}{2}-1)}]\mathrm{d}s$$
$$\lesssim \int_0^{\frac{t+K_2}{2}} [1 \wedge (t+K_2-s)^{-\frac{d}{2}}]\mathrm{d}s \lesssim (t+K_2)^{-(\frac{d}{2}-1)},$$

and

$$\int_{\frac{t+K_2}{2}}^{t+K_2} [1 \wedge (t+K_2-s)^{-\frac{d}{2}}][1 \wedge s^{-(\frac{d}{2}-1)}]\mathrm{d}s$$
$$\lesssim (t+K_2)^{-(\frac{d}{2}-1)} \int_{\frac{t+K_2}{2}}^{t+K_2} [1 \wedge (t+K_2-s)^{-\frac{d}{2}}]\mathrm{d}s \lesssim (t+K_2)^{-(\frac{d}{2}-1)},$$

which proves (3.8).

Now we iterate (3.7) to obtain $\gamma(t) \leq \sum_{n=0}^{\infty} \gamma_n(t)$ with

$$\gamma_0(t) = C\beta^2\alpha(t)$$
$$\gamma_n(t) = (C\beta^2)^{n+1} \int_{-K_2<s_n<\ldots<s_1<t} \prod_{j=0}^{n-1} k(s_j - s_{j+1})\alpha(s_n)\mathrm{d}s_n \ldots \mathrm{d}s_1$$

where we used the convention $s_0 = t$. From the explicit expression of α in (3.4), we know that there exists $C > 0$ such that

$$\alpha(s) \leq C[1 \wedge (s+K_2)^{-(\frac{d}{2}-1)}], \qquad \text{for } s > -K_2.$$

Now we apply (3.8) to derive that (with a possibly different constant $C > 0$)

$$\gamma_n(t) \leq (C\beta^2)^{n+1}[1 \wedge (t+K_2)^{-(\frac{d}{2}-1)}].$$

Choose $C\beta^2 < 1$ and sum over n, we know that

$$\gamma(t) \lesssim 1 \wedge (t+K_2)^{-(\frac{d}{2}-1)} \to 0$$

as $K_2 \to \infty$. The proof is complete. □

Remark 3.6. It is clear from the proof that the assumption of the constant initial data $u(0,x) \equiv u_K(-K,x) \equiv 1$ can be relaxed. A similar proof should work for more general initial conditions. One particular example for which our proof works is the following. Let $u(0,x)$ be a perturbation of the constant $\lambda > 0$ in the sense that $u(0,x) = \lambda + f(x)$ with $f \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, then (3.2) becomes

$$u_K(t,x) = \lambda + \int_{\mathbb{R}^d} p(t+K, x-y)f(y)\mathrm{d}y$$
$$+ \beta \int_{-K}^{t} \int_{\mathbb{R}^d} p(t-s, x-y)\sigma(u_K(s,y))\mathrm{d}W_\phi(s,y), \quad t > -K, x \in \mathbb{R}^d,$$

with the second term on the r.h.s., which is associated with the initial condition, depending on K as well. By following the same proof as before and the elementary fact that

$$\int_{\mathbb{R}^d} p(t+K, x-y)f(y)dy \lesssim 1 \wedge (t+K)^{-\frac{d}{2}},$$

we derive an integral inequality that is similar to (3.6) (recall that $K_1 > K_2 \geq 0$ and $t \geq -K_2$)

$$\tilde{\gamma}_{K_1,K_2}(t) \leq C[1 \wedge (t+K_2)^{-\frac{d}{2}}] + C\beta^2 \alpha_{K_1,K_2}(t) + C\beta^2 \int_{-K_2}^t [1 \wedge (t-s)^{-\frac{d}{2}}] \tilde{\gamma}_{K_1,K_2}(s) \mathrm{d}s$$

with $\tilde{\gamma}_{K_1,K_2}(t) := \sup_{x \in \mathbb{R}^d} \mathbb{E}[|u_{K_1}(t,x) - u_{K_2}(t,x)|^2]$. Using the fact that

$$1 \wedge (t+K_2)^{-\frac{d}{2}} \leq 1 \wedge (t+K_2)^{-(\frac{d}{2}-1)},$$

and applying Lemma 3.5 again, we conclude the proof.

Proof of Proposition 3.2. The tightness of $\{u(t,\cdot)\}_{t\geq 0}$ in $C(\mathbb{R}^d)$ follows from

(i) $\{u(t,0)\}_{t \geq 0}$ is tight in $\mathbb{R}$;

(ii) For any $\delta \in (0,1)$ and $n \in \mathbb{Z}_{\geq 0}$, there exists a constant $C > 0$ such that for $x_1, x_2 \in \mathbb{R}^d$ satisfying $|x_1 - x_2| \leq 1$ and any $t > 0$,

$$\mathbb{E}[|u(t,x_1) - u(t,x_2)|^{2n}] \leq C|x_1 - x_2|^{2\delta n}. \tag{3.9}$$

The tightness of $\{u(t,0)\}_{t\geq 0}$ comes from the bound $\sup_{t \geq 0} \|u(t,0)\|_p \leq C$ given by Lemma 2.1. To prove (3.9), we write

$$u(t,x_1) - u(t,x_2) = \beta \int_0^t \int_{\mathbb{R}^d} G_{x_1,x_2}(t-s,y)\sigma(u(s,y)) \mathrm{d}W_\phi(s,y).$$

with

$$G_{x_1,x_2}(t-s,y) = p(t-s,x_1-y) - p(t-s,x_2-y).$$

Follow the same argument in Lemma 2.1, we have

$$\begin{aligned}
&\mathbb{E}[|u(t,x_1) - u(t,x_2)|^{2n}] \\
&\lesssim \mathbb{E}\left[\left(\int_0^t \int_{\mathbb{R}^{2d}} G_{x_1,x_2}(t-s,y_1) G_{x_1,x_2}(t-s,y_2) \sigma(u(s,y_1))\sigma(u(s,y_2)) R(y_1-y_2) \mathrm{d}y_1 \mathrm{d}y_2 \mathrm{d}s\right)^n\right] \\
&\lesssim \left(\int_0^t \int_{\mathbb{R}^{2d}} G_{x_1,x_2}(t-s,y_1) G_{x_1,x_2}(t-s,y_2) \|\sigma(u(s,y_1))\sigma(u(s,y_2))\|_n R(y_1-y_2) \mathrm{d}y_1 \mathrm{d}y_2 \mathrm{d}s\right)^n \\
&\lesssim \left(\int_0^t \int_{\mathbb{R}^{2d}} G_{x_1,x_2}(t-s,y_1) G_{x_1,x_2}(t-s,y_2) R(y_1-y_2) \mathrm{d}y_1 \mathrm{d}y_2 \mathrm{d}s\right)^n.
\end{aligned}$$

By [5, Lemma 3.1], for any $\delta \in (0,1)$, there exists a constant $C > 0$ such that

$$G_{x_1,x_2}(t-s,y) \leq C(t-s)^{-\frac{\delta}{2}}[p(2(t-s),x_1-y) + p(2(t-s),x_2-y)]|x_1-x_2|^\delta.$$

Thus we can bound the integral as

$$\begin{aligned}
&\int_0^t \int_{\mathbb{R}^{2d}} G_{x_1,x_2}(t-s,y_1) G_{x_1,x_2}(t-s,y_2) R(y_1-y_2) \mathrm{d}y_1 \mathrm{d}y_2 \mathrm{d}s \\
&\lesssim |x_1-x_2|^{2\delta} \sum_{i,j=1,2} \int_0^t \int_{\mathbb{R}^{2d}} (t-s)^{-\delta} p(2(t-s),x_i-y_1) p(2(t-s),x_j-y_2) R(y_1-y_2) \mathrm{d}y_1 \mathrm{d}y_2 \mathrm{d}s.
\end{aligned}$$

For any i,j, we write the integral in Fourier domain to derive

$$\begin{aligned}
&\int_{\mathbb{R}^{2d}} p(2(t-s),x_i-y_1) p(2(t-s),x_j-y_2) R(y_1-y_2) \mathrm{d}y_1 \mathrm{d}y_2 \\
&= (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-4|\xi|^2(t-s)} \hat{R}(\xi) e^{i\xi\cdot(x_i-x_j)} \mathrm{d}\xi \lesssim \int_{\mathbb{R}^d} e^{-4|\xi|^2(t-s)} \hat{R}(\xi) \mathrm{d}\xi.
\end{aligned}$$

Another integration in s leads to

$$\int_0^t \int_{\mathbb{R}^d} (t-s)^{-\delta} e^{-4|\xi|^2(t-s)} \hat{R}(\xi) \mathrm{d}\xi \mathrm{d}s \leq \int_0^\infty s^{-\delta} e^{-4s} \mathrm{d}s \int_{\mathbb{R}^d} |\xi|^{2\delta-2} \hat{R}(\xi) \mathrm{d}\xi < \infty,$$

which implies

$$\mathbb{E}[|u(t,x_1) - u(t,x_2)|^{2n}] \leq C|x_1-x_2|^{2\delta n}$$

and completes the proof. □

4. Proof of Theorem 1.2

Recall that the goal is to show that for $g \in C_c^\infty(\mathbb{R}^d)$ and $t > 0$,

$$\frac{1}{\varepsilon^{\frac{d}{2}-1}} \int_{\mathbb{R}^d} (u_\varepsilon(t,x) - 1) g(x) \mathrm{d}x \Rightarrow \int_{\mathbb{R}^d} \mathcal{U}(t,x) g(x) \mathrm{d}x,$$

where $\mathcal{U}$ solves

$$\partial_t \mathcal{U} = \Delta \mathcal{U} + \beta \nu_\sigma \dot{W}(t,x). \tag{4.1}$$

Before going to the proof, we first give some heuristics which leads to the above equation and the expression of the effective variance

$$\nu_\sigma^2 = \int_{\mathbb{R}^d} \mathbb{E}[\sigma(Z(0))\sigma(Z(x))] R(x) \mathrm{d}x. \tag{4.2}$$

By the equation satisfied by u, we know that the diffusively rescaled solution satisfies

$$\partial_t u_\varepsilon = \Delta u_\varepsilon + \beta \varepsilon^{-2} \sigma(u_\varepsilon) \dot{W}_\phi(\tfrac{t}{\varepsilon^2}, \tfrac{x}{\varepsilon}).$$

Since $\dot{W}_\phi(t,x) = \int_{\mathbb{R}^d} \phi(x-y) \dot{W}(t,y) \mathrm{d}y$, using the scaling property of $\dot{W}$, we have

$$\varepsilon^{-2} \dot{W}_\phi(\tfrac{t}{\varepsilon^2}, \tfrac{x}{\varepsilon}) \overset{\text{law}}{=} \varepsilon^{\frac{d}{2}-1} \dot{W}_{\phi_\varepsilon}(t,x)$$

as random processes, with $\phi_\varepsilon(t,x) = \varepsilon^{-2-d} \phi(t/\varepsilon^2, x/\varepsilon)$ and

$$\dot{W}_{\phi_\varepsilon}(t,x) = \int_{\mathbb{R}^d} \phi_\varepsilon(x-y) \dot{W}(t,y) \mathrm{d}y.$$

Then the rescaled fluctuation has the same law as the solution to

$$\partial_t \left(\frac{u_\varepsilon - 1}{\varepsilon^{\frac{d}{2}-1}} \right) = \Delta \left(\frac{u_\varepsilon - 1}{\varepsilon^{\frac{d}{2}-1}} \right) + \beta \sigma(u_\varepsilon(t,x)) \dot{W}_{\phi_\varepsilon}(t,x). \tag{4.3}$$

By Theorem 1.1, for fixed $t > 0$, $\sigma(u_\varepsilon(t,x)) = \sigma(u(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}))$ has the same local statistical behavior as $\sigma(Z(\frac{x}{\varepsilon}))$ when $\varepsilon \ll 1$. Since the product between $\sigma(u_\varepsilon)$ and $\dot{W}_{\phi_\varepsilon}$ is in the Itô's sense, roughly speaking, these two terms are independent. The fact that $u(t,\cdot) \approx Z(\cdot)$ in law for microscopically large t induces a "renewal" mechanism which leads to a δ-correlation in time of the driving force $\sigma(u_\varepsilon(t,x))\dot{W}_{\phi_\varepsilon}(t,x)$ after passing to the limit. While the spatial covariance function of $\dot{W}_{\phi_\varepsilon}$ is $\varepsilon^{-d} R(\frac{\cdot}{\varepsilon})$, the overall spatial covariance function is

$$\begin{aligned}
&\mathbb{E}[\sigma(u_\varepsilon(t,x))\dot{W}_{\phi_\varepsilon}(t,x)\sigma(u_\varepsilon(t,y))\dot{W}_{\phi_\varepsilon}(t,y)] \\
\approx&\mathbb{E}[\sigma(Z(\tfrac{x}{\varepsilon}))\dot{W}_{\phi_\varepsilon}(t,x)\sigma(Z(\tfrac{y}{\varepsilon}))\dot{W}_{\phi_\varepsilon}(t,y)] \\
=&\mathbb{E}[\sigma(Z(\tfrac{x}{\varepsilon}))\sigma(Z(\tfrac{y}{\varepsilon}))]\mathbb{E}[\dot{W}_{\phi_\varepsilon}(t,x)\dot{W}_{\phi_\varepsilon}(t,y)] \\
=&\mathbb{E}[\sigma(Z(0))\sigma(Z(\tfrac{x-y}{\varepsilon}))] \cdot \varepsilon^{-d} R(\tfrac{x-y}{\varepsilon}).
\end{aligned}$$

After integrating the variable "$x-y$" out, we derive the effective variance in (4.2).

For $t > 0$ fixed, by the mild solution formulation

$$u_\varepsilon(t,x) = u\left(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}\right) = 1 + \beta \int_0^{\frac{t}{\varepsilon^2}} \int_{\mathbb{R}^d} p\left(\frac{t}{\varepsilon^2} - s, \frac{x}{\varepsilon} - y\right) \sigma(u(s,y)) \mathrm{d}W_\phi(s,y),$$

we may write

$$\begin{aligned}
X_\varepsilon =& \frac{1}{\varepsilon^{\frac{d}{2}-1}} \int_{\mathbb{R}^d} (u_\varepsilon(t,x) - 1) g(x) \mathrm{d}x \\
=& \frac{\beta}{\varepsilon^{\frac{d}{2}-1}} \int_{\mathbb{R}^d} \left(\int_0^{\frac{t}{\varepsilon^2}} \int_{\mathbb{R}^d} p\left(\frac{t}{\varepsilon^2} - s, \frac{x}{\varepsilon} - y\right) \sigma(u(s,y)) \mathrm{d}W_\phi(s,y) \right) g(x) \mathrm{d}x \\
=& \delta(v_\varepsilon),
\end{aligned} \tag{4.4}$$

where we recall that $\delta(\cdot)$ is the divergence operator defined in (2.2) and

$$v_\varepsilon(s,y) = \frac{\beta}{\varepsilon^{\frac{d}{2}-1}} \mathbb{1}_{[0,\frac{t}{\varepsilon^2}]}(s)\sigma(u(s,y)) \int_{\mathbb{R}^d} p\left(\frac{t}{\varepsilon^2}-s, \frac{x}{\varepsilon}-y\right) g(x)\mathrm{d}x.$$

As $\mathcal{U}$ solves the equation $\partial_t \mathcal{U} = \Delta \mathcal{U} + \beta\nu_\sigma \dot{W}(t,x)$, we have $\int_{\mathbb{R}^d} \mathcal{U}(t,x)g(x)\mathrm{d}x$ is of Gaussian distribution with zero mean and variance

$$\Sigma_g := \mathrm{Var}\left[\int_{\mathbb{R}^d} \mathcal{U}(t,x)g(x)\mathrm{d}x\right] = \beta^2\nu_\sigma^2 \int_0^t \int_{\mathbb{R}^{2d}} p(2(t-s), x_1-x_2)g(x_1)g(x_2)\mathrm{d}x_1\mathrm{d}x_2.$$

Thus, the proof of Theorem 1.2 reduces to showing that

$$X_\varepsilon = \delta(v_\varepsilon) \Rightarrow N(0,\Sigma_g), \qquad \text{as } \varepsilon \to 0.$$

By Proposition 2.3, we only need to show

$$\mathbb{E}[|\Sigma_g - \langle DX_\varepsilon, v_\varepsilon\rangle_{\mathcal{H}}|^2] \to 0, \qquad \text{as } \varepsilon \to 0. \tag{4.5}$$

From (4.4), the Malliavin derivative of X_ε satisfies

$$\begin{aligned} D_{s,y}X_\varepsilon =& v_\varepsilon(s,y) + \frac{\beta}{\varepsilon^{\frac{d}{2}-1}} \int_{\mathbb{R}^d} \left(\int_s^{\frac{t}{\varepsilon^2}} \int_{\mathbb{R}^d} p\left(\frac{t}{\varepsilon^2}-r, \frac{x}{\varepsilon}-z\right) D_{s,y}\sigma(u(r,z))\mathrm{d}W_\phi(r,z)\right) g(x)\mathrm{d}x \\ =& v_\varepsilon(s,y) + \frac{\beta}{\varepsilon^{\frac{d}{2}-1}} \int_{\mathbb{R}^d} \left(\int_s^{\frac{t}{\varepsilon^2}} \int_{\mathbb{R}^d} p\left(\frac{t}{\varepsilon^2}-r, \frac{x}{\varepsilon}-z\right) \Sigma(r,z)D_{s,y}u(r,z)\mathrm{d}W_\phi(r,z)\right) g(x)\mathrm{d}x, \end{aligned}$$

and $\Sigma(r,z)$ as a random variable is bounded by the Lipschitz constant σ_{Lip}. Recall that

$$\langle h,g\rangle_{\mathcal{H}} = \int_{\mathbb{R}^{1+2d}} h(s,x)g(s,y)R(x-y)\mathrm{d}x\mathrm{d}y\mathrm{d}s$$

for all $h, g \in \mathcal{H}$, we have

$$\langle DX_\varepsilon, v_\varepsilon\rangle_{\mathcal{H}} = \frac{\beta^2}{\varepsilon^{d-2}}(A_{1,\varepsilon} + A_{2,\varepsilon}),$$

where

(4.6)

$$\begin{aligned} A_{1,\varepsilon} =& \int_0^{\frac{t}{\varepsilon^2}} \int_{\mathbb{R}^{2d}} \left(\int_{\mathbb{R}^d} p\left(\frac{t}{\varepsilon^2}-s, \frac{x_1}{\varepsilon}-y_1\right) g(x_1)\mathrm{d}x_1\right) \\ &\cdot \left(\int_{\mathbb{R}^d} p\left(\frac{t}{\varepsilon^2}-s, \frac{x_2}{\varepsilon}-y_2\right) g(x_2)\mathrm{d}x_2\right) \sigma(u(s,y_1))\sigma(u(s,y_2))R(y_1-y_2)\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}s, \end{aligned}$$

and

(4.7)

$$\begin{aligned} A_{2,\varepsilon} =& \int_0^{\frac{t}{\varepsilon^2}} \int_{\mathbb{R}^{2d}} \left(\int_s^{\frac{t}{\varepsilon^2}} \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} p\left(\frac{t}{\varepsilon^2}-r, \frac{x_1}{\varepsilon}-z\right) g(x_1)\mathrm{d}x_1\right) \Sigma(r,z)D_{s,y_1}u(r,z)\mathrm{d}W_\phi(r,z)\right) \\ &\cdot \left(\int_{\mathbb{R}^d} p\left(\frac{t}{\varepsilon^2}-s, \frac{x_2}{\varepsilon}-y_2\right) g(x_2)\mathrm{d}x_2\right) \sigma(u(s,y_2))R(y_1-y_2)\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}s. \end{aligned}$$

We also notice that

$$\begin{aligned} \mathbb{E}[|\Sigma_g - \langle DX_\varepsilon, v_\varepsilon\rangle_{\mathcal{H}}|^2] &= \mathbb{E}[|\Sigma_g - \beta^2\varepsilon^{2-d}(A_{1,\varepsilon} + A_{2,\varepsilon})|^2] \\ &\le 2\|\Sigma_g - \beta^2\varepsilon^{2-d}A_{1,\varepsilon}\|_2^2 + 2\beta^4\varepsilon^{4-2d}\|A_{2,\varepsilon}\|_2^2, \end{aligned}$$

so to complete the proof of Theorem 1.2, it remains to show the right-hand side of the above inequality goes to zero as $\varepsilon \to 0$.

Lemma 4.1. *As* $\varepsilon \to 0$, $\|\Sigma_g - \beta^2\varepsilon^{2-d}A_{1,\varepsilon}\|_2 \to 0$.

Lemma 4.2. *As* $\varepsilon \to 0$, $\varepsilon^{2-d}\|A_{2,\varepsilon}\|_2 \to 0$.

In the proofs of Lemma 4.1 and 4.2, we will use the notation

$$g_t(x) = \int_{\mathbb{R}^d} p(t, x-y) g(y) \mathrm{d}y, \qquad t > 0, x \in \mathbb{R}^d,$$

so

$$|g_t(x)| \leq \|g\|_{L^\infty(\mathbb{R}^d)}, \qquad \int_{\mathbb{R}^d} |g_t(x)| \mathrm{d}x \leq \|g\|_{L^1(\mathbb{R}^d)},$$

for all $t > 0, x \in \mathbb{R}^d$. Without loss of generality, we assume the function g is non-negative when we estimate integrals involving g.

Proof of Lemma 4.1. We first simplify the expression of $A_{1,\varepsilon}$ defined in (4.6). After the change of variables $y_1 \mapsto y_1/\varepsilon, y_2 \mapsto y_2/\varepsilon, s \mapsto s/\varepsilon^2$ and use the scaling property of the heat kernel, we have

(4.8)

$$\begin{aligned} A_{1,\varepsilon} =& \varepsilon^{-2} \int_0^t \int_{\mathbb{R}^{2d}} \left(\int_{\mathbb{R}^d} p(t-s, x_1 - y_1) g(x_1) \mathrm{d}x_1 \right) \\ & \cdot \left(\int_{\mathbb{R}^d} p(t-s, x_2 - y_2) g(x_2) \mathrm{d}x_2 \right) \sigma\left(u\left(\frac{s}{\varepsilon^2}, \frac{y_1}{\varepsilon}\right)\right) \sigma\left(u\left(\frac{s}{\varepsilon^2}, \frac{y_2}{\varepsilon}\right)\right) R\left(\frac{y_1 - y_2}{\varepsilon}\right) \mathrm{d}y_1 \mathrm{d}y_2 \mathrm{d}s. \end{aligned}$$

Further change $\frac{y_1 - y_2}{\varepsilon} \mapsto z$ and $y_2 \mapsto y$, we obtain

$$\varepsilon^{2-d} A_{1,\varepsilon} = \int_0^t \int_{\mathbb{R}^{2d}} g_{t-s}(y + \varepsilon z) g_{t-s}(y) \sigma\left(u\left(\frac{s}{\varepsilon^2}, \frac{y}{\varepsilon} + z\right)\right) \sigma\left(u\left(\frac{s}{\varepsilon^2}, \frac{y}{\varepsilon}\right)\right) R(z) \mathrm{d}y \mathrm{d}z \mathrm{d}s,$$

where we recall that $g_{t-s}(y) = \int_{\mathbb{R}^d} p(t-s, y-z) g(z) \mathrm{d}z$. The proof is then divided into two steps:

(i) $\beta^2 \varepsilon^{2-d} \mathbb{E}[A_{1,\varepsilon}] \to \Sigma_g$ as $\varepsilon \to 0$.

(ii) $\varepsilon^{4-2d} \mathrm{Var}[A_{1,\varepsilon}] \to 0$ as $\varepsilon \to 0$.

To prove (i), it suffices to note that $u(s/\varepsilon^2, x)$ is stationary in x−variable, so

$$\beta^2 \varepsilon^{2-d} \mathbb{E}[A_{1,\varepsilon}] = \beta^2 \int_0^t \int_{\mathbb{R}^{2d}} g_{t-s}(y + \varepsilon z) g_{t-s}(y) \mathbb{E}\left[\sigma\left(u\left(\frac{s}{\varepsilon^2}, z\right)\right) \sigma\left(u\left(\frac{s}{\varepsilon^2}, 0\right)\right)\right] R(z) \mathrm{d}y \mathrm{d}z \mathrm{d}s.$$

By Theorem 1.1, we know that for $s > 0, z \in \mathbb{R}^d$, the random vector

$$(u(s/\varepsilon^2, z), u(s/\varepsilon^2, 0)) \Rightarrow (Z(z), Z(0))$$

in distribution as $\varepsilon \to 0$. By the fact that σ is Lipschitz and applying Lemma 2.1, we have the uniform integrability to pass to the limit and conclude that

$$\beta^2 \varepsilon^{2-d} \mathbb{E}[A_{1,\varepsilon}] \to \beta^2 \int_0^t \int_{\mathbb{R}^{2d}} |g_{t-s}(y)|^2 \mathbb{E}[\sigma(Z(z)) \sigma(Z(0))] R(z) \mathrm{d}y \mathrm{d}z \mathrm{d}s = \Sigma_g.$$

To prove (ii), we first use (4.8) to write

(4.9)

$$\begin{aligned} \varepsilon^{4-2d} \mathrm{Var}[A_{1,\varepsilon}] =& \varepsilon^{-2d} \int_0^t \int_{\mathbb{R}^{4d}} g_{t-s}(y_1) g_{t-s}(y_2) g_{t-s}(y_1') g_{t-s}(y_2') \\ & \cdot \mathrm{Cov}[\Lambda_\varepsilon(s, y_1, y_2), \Lambda_\varepsilon(s, y_1', y_2')] R\left(\frac{y_1 - y_2}{\varepsilon}\right) R\left(\frac{y_1' - y_2'}{\varepsilon}\right) \mathrm{d}y_1 \mathrm{d}y_2 \mathrm{d}y_1' \mathrm{d}y_2' \mathrm{d}s, \end{aligned}$$

where

$$\Lambda_\varepsilon(s, y_1, y_2) = \sigma\left(u\left(\frac{s}{\varepsilon^2}, \frac{y_1}{\varepsilon}\right)\right) \sigma\left(u\left(\frac{s}{\varepsilon^2}, \frac{y_2}{\varepsilon}\right)\right).$$

Applying the Clark-Ocone formula (Proposition 2.4) to Λ_ε, we obtain that

$$\Lambda_\varepsilon(s, y_1, y_2) = \mathbb{E}[\Lambda_\varepsilon(s, y_1, y_2)] + \int_0^{\frac{s}{\varepsilon^2}} \int_{\mathbb{R}^d} \mathbb{E}[D_{r,z} \Lambda_\varepsilon(s, y_1, y_2) | \mathscr{F}_r] \mathrm{d}W_\phi(r, z),$$

from which we deduce that

$$\mathrm{Cov}[\Lambda_\varepsilon(s,y_1,y_2),\Lambda_\varepsilon(s,y_1',y_2')]=\int_0^{\frac{s}{\varepsilon^2}}\int_{\mathbb{R}^{2d}}\mathbb{E}\left[\mathbb{E}[D_{r,z_1}\Lambda_\varepsilon(s,y_1,y_2)|\mathscr{F}_r]\mathbb{E}[D_{r,z_2}\Lambda_\varepsilon(s,y_1',y_2')|\mathscr{F}_r]\right]\cdot R(z_1-z_2)\mathrm{d}z_1\mathrm{d}z_2\mathrm{d}r.$$

By the Chain Rule, we have

$$\begin{aligned}D_{r,z}\Lambda_\varepsilon(s,y_1,y_2)=&\Sigma\left(\frac{s}{\varepsilon^2},\frac{y_1}{\varepsilon}\right)D_{r,z}u\left(\frac{s}{\varepsilon^2},\frac{y_1}{\varepsilon}\right)\sigma\left(u\left(\frac{s}{\varepsilon^2},\frac{y_2}{\varepsilon}\right)\right)\\&+\Sigma\left(\frac{s}{\varepsilon^2},\frac{y_2}{\varepsilon}\right)D_{r,z}u\left(\frac{s}{\varepsilon^2},\frac{y_2}{\varepsilon}\right)\sigma\left(u\left(\frac{s}{\varepsilon^2},\frac{y_1}{\varepsilon}\right)\right).\end{aligned}$$

Applying Lemma 2.1, 2.2, and using the fact that Σ is uniformly bounded, we derive that

$$\begin{aligned}\|\mathbb{E}[D_{r,z}\Lambda_\varepsilon(s,y_1,y_2)|\mathscr{F}_r]\|_2\le&\|D_{r,z}\Lambda_\varepsilon(s,y_1,y_2)\|_2\\\lesssim&p\left(\frac{s}{\varepsilon^2}-r,\frac{y_1}{\varepsilon}-z_1\right)+p\left(\frac{s}{\varepsilon^2}-r,\frac{y_2}{\varepsilon}-z_1\right).\end{aligned}$$

Therefore, we deduce that

$$\begin{aligned}&\left|\mathrm{Cov}[\Lambda_\varepsilon(s,y_1,y_2),\Lambda_\varepsilon(s,y_1',y_2')]\right|\\&\lesssim\int_0^{\frac{s}{\varepsilon^2}}\int_{\mathbb{R}^{2d}}\left(p\left(\frac{s}{\varepsilon^2}-r,\frac{y_1}{\varepsilon}-z_1\right)+p\left(\frac{s}{\varepsilon^2}-r,\frac{y_2}{\varepsilon}-z_1\right)\right)\\&\quad\cdot\left(p\left(\frac{s}{\varepsilon^2}-r,\frac{y_1'}{\varepsilon}-z_2\right)+p\left(\frac{s}{\varepsilon^2}-r,\frac{y_2'}{\varepsilon}-z_2\right)\right)R(z_1-z_2)\mathrm{d}z_1\mathrm{d}z_2\mathrm{d}r\\&=\sum_{i,j=1,2}F\left(\frac{s}{\varepsilon^2},\frac{y_i-y_j'}{\varepsilon}\right),\end{aligned}$$

where

$$\begin{aligned}F\left(\frac{s}{\varepsilon^2},\frac{y_i-y_j'}{\varepsilon}\right)=&\int_0^{\frac{s}{\varepsilon^2}}\int_{\mathbb{R}^{2d}}p\left(r,\frac{y_i-y_j'}{\varepsilon}-z_1\right)p\left(r,z_2\right)R(z_1-z_2)\mathrm{d}z_1\mathrm{d}z_2\mathrm{d}r\\\le&\tilde{F}\left(\frac{y_i-y_j'}{\varepsilon}\right),\end{aligned}$$

with

$$\tilde{F}(x):=\int_0^\infty\int_{\mathbb{R}^{2d}}p(r,x-z_1)p(r,z_2)R(z_1-z_2)\mathrm{d}z_1\mathrm{d}z_2\mathrm{d}r\lesssim 1\wedge|x|^{2-d}.$$

Going back to (4.9), it suffices to estimate the integral

$$\begin{aligned}&\varepsilon^{-2d}\int_0^t\int_{\mathbb{R}^{4d}}g_{t-s}(y_1)g_{t-s}(y_2)g_{t-s}(y_1')g_{t-s}(y_2')\\&\qquad\cdot\tilde{F}\left(\frac{y_i-y_j'}{\varepsilon}\right)R\left(\frac{y_1-y_2}{\varepsilon}\right)R\left(\frac{y_1'-y_2'}{\varepsilon}\right)\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}y_1'\mathrm{d}y_2'\mathrm{d}s\end{aligned}$$

for $i,j=1,2$ and show it vanishes as $\varepsilon\to0$. By symmetry, we only need to consider the case $i=j=1$. After a change of variables $y_1\mapsto y_2+\varepsilon y_1, y_1'\mapsto y_2'+\varepsilon y_1'$ and use the fact that $|g_{t-s}(\cdot)|\le\|g\|_\infty$, the above expression is bounded by

$$\begin{aligned}&\int_0^t\int_{\mathbb{R}^{4d}}g_{t-s}(y_2)g_{t-s}(y_2')R(y_1)R(y_1')\tilde{F}\left(\frac{y_2-y_2'}{\varepsilon}+y_1-y_1'\right)\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}y_1'\mathrm{d}y_2'\mathrm{d}s\\&\lesssim\varepsilon^{d-2}\int_0^t\int_{\mathbb{R}^{4d}}g_{t-s}(y_2)g_{t-s}(y_2')R(y_1)R(y_1')|y_2-y_2'|^{2-d}\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}y_1'\mathrm{d}y_2'\mathrm{d}s\lesssim\varepsilon^{d-2}.\end{aligned}$$

The proof is complete. □

Proof of Lemma 4.2. Define

$$\tilde{A}_{2,\varepsilon}(s,x_1,y_1) := \int_s^{\frac{t}{\varepsilon^2}} \int_{\mathbb{R}^d} p\left(\frac{t}{\varepsilon^2} - r, \frac{x_1}{\varepsilon} - z\right) \Sigma(r,z) D_{s,y_1} u(r,z) \mathrm{d}W_\phi(r,z), \tag{4.10}$$

and we can write

$$A_{2,\varepsilon} = \int_0^{\frac{t}{\varepsilon^2}} \int_{\mathbb{R}^{4d}} \tilde{A}_{2,\varepsilon}(s,x_1,y_1) p\left(\frac{t}{\varepsilon^2} - s, \frac{x_2}{\varepsilon} - y_2\right) \sigma(u(s,y_2)) \cdot g(x_1) g(x_2) R(y_1 - y_2) \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}y_1 \mathrm{d}y_2 \mathrm{d}s.$$

After the change of variable $s \mapsto s/\varepsilon^2, y_i \mapsto y_i/\varepsilon$ and integrating in x_2, we have

$$A_{2,\varepsilon} = \varepsilon^{-d-2} \int_0^t \int_{\mathbb{R}^{3d}} \tilde{A}_{2,\varepsilon}\left(\frac{s}{\varepsilon^2}, x_1, \frac{y_1}{\varepsilon}\right) \sigma\left(u\left(\frac{s}{\varepsilon^2}, \frac{y_2}{\varepsilon}\right)\right) g(x_1) g_{t-s}(y_2) \cdot R\left(\frac{y_1 - y_2}{\varepsilon}\right) \mathrm{d}x_1 \mathrm{d}y_1 \mathrm{d}y_2 \mathrm{d}s.$$

Define

$$B_\varepsilon(s,x_1,x_1',y_1,y_1',y_2,y_2') = \mathbb{E}\left[\tilde{A}_{2,\varepsilon}\left(\frac{s}{\varepsilon^2}, x_1, \frac{y_1}{\varepsilon}\right) \tilde{A}_{2,\varepsilon}\left(\frac{s}{\varepsilon^2}, x_1', \frac{y_1'}{\varepsilon}\right) \sigma\left(u\left(\frac{s}{\varepsilon^2}, \frac{y_2}{\varepsilon}\right)\right) \sigma\left(u\left(\frac{s}{\varepsilon^2}, \frac{y_2'}{\varepsilon}\right)\right)\right],$$

then by Minkowski inequality, we have

$$\|A_{2,\varepsilon}\|_2 \leq \varepsilon^{-d-2} \int_0^t \left(\int_{\mathbb{R}^{6d}} B_\varepsilon(s,x_1,x_1',y_1,y_1',y_2,y_2') g(x_1) g(x_1') g_{t-s}(y_2) g_{t-s}(y_2') \cdot R\left(\frac{y_1 - y_2}{\varepsilon}\right) R\left(\frac{y_1' - y_2'}{\varepsilon}\right) \mathrm{d}x_1 \mathrm{d}x_1' \mathrm{d}y_1 \mathrm{d}y_2 \mathrm{d}y_1' \mathrm{d}y_2'\right)^{\frac{1}{2}} \mathrm{d}s. \tag{4.11}$$

Meanwhile, by the expression of $\tilde{A}_{2,\varepsilon}$ in (4.10) and Itô's isometry, we have

$$B_\varepsilon(s,x_1,x_1',y_1,y_1',y_2,y_2') = \int_{\frac{s}{\varepsilon^2}}^{\frac{t}{\varepsilon^2}} \int_{\mathbb{R}^{2d}} \mathbb{E}\left[\Sigma(r,z_1)\Sigma(r,z_2) D_{\frac{s}{\varepsilon^2},\frac{y_1}{\varepsilon}} u(r,z_1) D_{\frac{s}{\varepsilon^2},\frac{y_1'}{\varepsilon}} u(r,z_2) \sigma\left(u\left(\frac{s}{\varepsilon^2}, \frac{y_2}{\varepsilon}\right)\right) \sigma\left(u\left(\frac{s}{\varepsilon^2}, \frac{y_2'}{\varepsilon}\right)\right)\right] \cdot p\left(\frac{t}{\varepsilon} - r, \frac{x_1}{\varepsilon} - z_1\right) p\left(\frac{t}{\varepsilon} - r, \frac{x_1'}{\varepsilon} - z_2\right) R(z_1 - z_2) \mathrm{d}z_1 \mathrm{d}z_2 \mathrm{d}r.$$

Applying Lemma 2.1, 2.2, Cauchy-Schwarz inequality and a change of variables,

$$|B_\varepsilon(s,x_1,x_1',y_1,y_1',y_2,y_2')| \lesssim \varepsilon^{2d-2} \int_s^t \int_{\mathbb{R}^{2d}} p(r-s, z_1 - y_1) p(r-s, z_2 - y_1') p(t-r, x_1 - z_1) \cdot p(t-r, x_1' - z_2) R\left(\frac{z_1 - z_2}{\varepsilon}\right) \mathrm{d}z_1 \mathrm{d}z_2 \mathrm{d}r.$$

Substitute the above estimate into (4.11) and integrate in x_1, x_1', we finally obtain

$$\|A_{2,\varepsilon}\|_2 \lesssim \varepsilon^{-3} \int_0^t \left(\int_s^t \int_{\mathbb{R}^{6d}} p(r-s, z_1 - y_1) p(r-s, z_2 - y_1') g_{t-r}(z_1) g_{t-r}(z_2) g_{t-s}(y_2) g_{t-s}(y_2') \cdot R\left(\frac{y_1 - y_2}{\varepsilon}\right) R\left(\frac{y_1' - y_2'}{\varepsilon}\right) R\left(\frac{z_1 - z_2}{\varepsilon}\right) \mathrm{d}y_1 \mathrm{d}y_2 \mathrm{d}y_1' \mathrm{d}y_2' \mathrm{d}z_1 \mathrm{d}z_2 \mathrm{d}r\right)^{\frac{1}{2}} \mathrm{d}s.$$

For the integral on the right-hand side of the above inequality, we compute the integral in y_1, y_2 explicitly:

$$\int_{\mathbb{R}^{2d}} p(r-s, z_1 - y_1) g_{t-s}(y_2) R\left(\frac{y_1 - y_2}{\varepsilon}\right) \mathrm{d}y_1 \mathrm{d}y_2 = \varepsilon^d \int_{\mathbb{R}^d} g_{t+r-2s}(z_1 - \varepsilon y_1) R(y_1) \mathrm{d}y_1 \lesssim \varepsilon^d.$$

Similarly, the integral in y_1', y_2' is also bounded by

$$\int_{\mathbb{R}^{2d}} p(r-s, z_2 - y_1') g_{t-s}(y_2') R\left(\frac{y_1' - y_2'}{\varepsilon}\right) \mathrm{d}y_1' \mathrm{d}y_2' \lesssim \varepsilon^d.$$

Thus,

$$\|A_{2,\varepsilon}\|_2 \lesssim \varepsilon^{d-3} \int_0^t \left(\int_s^t \int_{\mathbb{R}^{2d}} g_{t-r}(z_1) g_{t-r}(z_2) R\left(\frac{z_1 - z_2}{\varepsilon}\right) \mathrm{d}z_1 \mathrm{d}z_2 \mathrm{d}r \right)^{\frac{1}{2}} \mathrm{d}s \lesssim \varepsilon^{\frac{3d}{2}-3}.$$

The proof is complete. □

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