Fluctuations of random semi-linear advection equations

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Abstract

We consider a semi-linear advection equation driven by a highly-oscillatory space-time Gaussian random field, with the randomness affecting both the drift and the nonlinearity. In the linear setting, classical results show that the characteristics converge in distribution to a homogenized Brownian motion, hence the point-wise law of the solution is close to a functional of the Brownian motion. Our main result is that the nonlinearity plays the role of a *random diffeomorphism*, and the point-wise limiting distribution is obtained by applying the diffeomorphism to the limit in the linear setting.

1 Introduction

In this paper, we consider solutions to the semi-linear advection equations with rapidly oscillating random coefficients, of the form

$$\begin{aligned}
&\partial_t u_\varepsilon(t,x) + \frac{1}{\varepsilon} V\Big(\frac{t}{\varepsilon^2},\frac{x}{\varepsilon}\Big)\cdot\nabla_x u_\varepsilon(t,x) = \frac{1}{\varepsilon^\alpha} f\Big(t,x,u_\varepsilon(t,x),V\Big(\frac{t}{\varepsilon^2},\frac{x}{\varepsilon}+\cdot\Big)\Big),\\
&u(T,x)=u_0(x),\quad t<T, x\in\mathbb{R}^d.
\end{aligned}\tag{1.1}$$

Here, $V(t,x)$ is a zero-mean, incompressible, stationary Gaussian, vector-valued random field, and the nonlinear term f depends on both u_ε and V. The parameter $\alpha\geq 0$ is to be chosen so that the nonlinearity plays a non-trivial role as $\varepsilon\to 0$. The linear problem with $f\equiv 0$ corresponds to the *passive scalar* model that describes a particle drifting in a time-dependent, incompressible random environment and has applications in both turbulent diffusion and stochastic homogenization, see e.g. [23, 34, 31] and the references therein. The model has been extensively studied, both in the mathematics and physics literature, under various assumptions on the advection $V(t,x)$. A typical result shows that the underlying characteristics, i.e., the trajectory of the particle, converge to a diffusion, see e.g. [4, 5, 7, 11, 15, 18]. The problem may also exhibit a memory effect if the space-time correlations of $V(t,x)$ decay sufficiently slowly so that the corresponding trajectory process converges to a non-Markovian limit, see [1, 12, 17, 19].

Homogenization problems for quasilinear, stochastic Hamilton-Jacobi type equations, with or without the presence of a viscous term, were extensively studied in the periodic, almost periodic and ergodic settings, starting with the work of [25], see also [6, 33, 30, 22, 21, 2] and the references therein. These problems usually involve the hyperbolic scaling, i.e., the time and spatial variables are scaled as $t/\varepsilon, x/\varepsilon$, compared with the diffusive scaling appearing in (1.1). Concerning the diffusive scaling, the question of homogenization for some classes of semi-linear (or even of quasi-linear)

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parabolic equations with periodic, or ergodc coefficients, using backward stochastic differential equation techniques, has been considered in [15, 8, 27, 29, 9].

In this paper, we stay in the regime where V decorrelates fast, and our goal is to understand the interaction between the randomness and the nonlinearity, and the asymptotic behavior of u_ε, as well as the multi-point statistics $u_\varepsilon(t,x_1),\ldots,u_\varepsilon(t,x_N)$ for any number of points $(t,x_1),\ldots,(t,x_N)$.

As we have mentioned, when $f\equiv 0$, the equation (1.1) is a classical problem of a passive scalar in an evolving random environment, and the solution can be expressed as

$$u_\varepsilon(t,x)=u_0(X_\varepsilon^{t,x}(T)). \tag{1.2}$$

Here, $X_\varepsilon^{t,x}(\cdot)$ is the characteristic of (1.1) starting from (t,x):

$$\frac{d}{ds}X_\varepsilon^{t,x}(s)=\frac{1}{\varepsilon}V\Big(\frac{s}{\varepsilon^2},\frac{X_\varepsilon^{t,x}(s)}{\varepsilon}\Big),\ s>t;\quad X_\varepsilon^{t,x}(t)=x. \tag{1.3}$$

It was shown in, e.g., [7, 11, 20] that the process $(X_\varepsilon^{t,x}(s))_{s\geq t}$ converges in law to $(x+\beta_{s-t})_{s\geq t}$. Here, $(\beta_t)_{t\geq 0}$ is a Brownian motion with a covariance matrix that can be computed through the statistics of V, see (4.10) below. As a result, for fixed (t,x), $u_\varepsilon(t,x)$ converges in distribution to $u_0(x+\beta_{T-t})$. For two different starting points $x_1\neq x_2$, the trajectories X_ε^{t,x_1} and X_ε^{t,x_2} experience random environments that are typically at distances of order $O(1/\varepsilon)$ away from each other on the microscopic spatial scale. As a result, the two trajectories become nearly independent, when $\varepsilon\to 0$, provided that the velocity field $V(t,x)$ decorrelates fast in space. This happens even if the spatial realizations of V are analytic, which precludes the spatial mixing of the field. Similarly, for an arbitrary number of initial starting points, the random vector $(u_\varepsilon(t,x_1),\ldots,u_\varepsilon(t,x_N))$ converges in law to $(u_0(x_1+\beta^{(1)}_{T-t}),\ldots,u_0(x_N+\beta^{(N)}_{T-t}))$, where $(\beta^{(j)}_t)_{t\geq 0}$ are i.i.d. copies of the effective Brownian motion, see Theorem 2.1 below. In particular, the above result implies that, after averaging in space (i.e. taking the weak spatial limit), the randomness averages out and the limit becomes deterministic. More precisely we have

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^d}u_\varepsilon(t,x)g(x)dx=\int_{\mathbb{R}^d}\mathbb{E}[u_\varepsilon(t,x)]g(x)dx\qquad\text{for } g\in L^1(\mathbb{R}^d), \tag{1.4}$$

see Corollary 2.2. This remains in sharp contrast with the parabolic setting (see [15]), where both the point-wise limit and the limit measured weakly in the spatial variable are both deterministic.

In the non-linear setting, when $f\neq 0$, the solution along the characteristics is not constant but rather satisfies

$$u_\varepsilon(s,X_\varepsilon^{t,x}(s))+\frac{1}{\varepsilon^\alpha}\int_s^T f\Big(\sigma,X_\varepsilon^{t,x}(\sigma),u_\varepsilon(\sigma,X_\varepsilon^{t,x}(\sigma)),V(\frac{\sigma}{\varepsilon^2},\frac{X_\varepsilon^{t,x}(\sigma)}{\varepsilon}+\cdot)\Big)d\sigma=u_0(X_\varepsilon^{t,x}(T)),\quad s\in[t,T]. \tag{1.5}$$

If the nonlinearity has a non-zero mean $\bar f=\mathbb{E}[f]$, we can roughly treat it as deterministic to the leading order, in light of the averaging induced by the V variable in (1.5). This leads to the choice $\alpha=0$. Replacing $f\to\bar f$, we obtain from (1.5):

$$u_\varepsilon(s,X_\varepsilon^{t,x}(s))+\int_s^T\bar f\Big(\sigma,X_\varepsilon^{t,x}(\sigma),u_\varepsilon(\sigma,X_\varepsilon^{t,x}(\sigma))\Big)d\sigma=u_0(X_\varepsilon^{t,x}(T)),\quad s\in[t,T], \tag{1.6}$$

a "deterministic" integral equation in time, driven by the random charateristics. Since $X_\varepsilon^{t,x}$ converges to the effective Brownian motion, it is not hard to see from (1.6), at least formally, that $u_\varepsilon(s,X_\varepsilon^{t,x}(s))$ converges to the solution $\mathcal{U}(t,x)$ of an integral equation driven by the effective Brownian motion.

This argument can be also extended to arbitrary points $x_1, \dots, x_N$, showing that random vectors $(u_\varepsilon(t,x_1),\dots,u_\varepsilon(t,x_N))$ converge in law to $\Big(\mathcal{U}^{(1)}(t,x_1),\dots,\mathcal{U}^{(N)}(t,x_N)\Big)$, where $\mathcal{U}^{(j)}(t,x)$ correspond to solutions driven by independent copies of the effective Brownian motion. If the fluctuation is measured weakly-in-space, it can also be shown that (1.4) holds. The precise statement of the results can be found in Theorem 2.3 and Corollary 2.4.

When $\bar{f}=0$, the random effect of f comes up in the next order, and the standard central limit scaling suggests the choice $\alpha=1$. Due to the interaction between the two random sources, $X_\varepsilon^{t,x}$ and V, the asymptotic behavior of the integral

$$\frac{1}{\varepsilon}\int_s^T f\Big(\sigma, X_\varepsilon^{t,x}(\sigma), u_\varepsilon(\sigma, X_\varepsilon^{t,x}(\sigma)), V\big(\frac{\sigma}{\varepsilon^2}, \frac{X_\varepsilon^{t,x}(\sigma)}{\varepsilon}+\cdot\big)\Big)d\sigma$$

that appears in (1.5) is much more complicated than that in (1.6). We will in this case obtain in the limit a "random" integral equation driving by the effective Brownian motion. This is the objective of Theorem 2.6.

Let us briefly describe the principal ingredients of the proofs of our main results and organization of the paper. Section 2 contains the main results of this paper, and the assumptions on the random advection $V(t,x)$. The analysis of the solutions of semi-linear advection equations is based on the method of characteristics that translates the asymptotics of u_ε into the study of the random spatial trajectories, together with the evolution of u_ε along the characteristics, described by (1.5), together with the inverse of the corresponding flow map coming from (1.5). An important tool in this approach is the process that describes the random velocity V along the spatial characteristics – the so-called environment process, see Section 4. The main technical novelty in the analysis here is the approach to the analysis of the environment process. It is shown in Section 3 that the Gaussian velocity fields, considered in the present paper, are actually Markovian. Fields of this type appeared quite frequently throughout the literature, see e.g. [16, Chapter 12] and the references therein. What is novel in our present approach, compared with that of [16], is the use of the respective Cameron-Martin space in the description of the dynamics of the field, see Section 3. It allows us to find a simple semimartingale representation of the dynamics, see the stochastic differential equation (3.20), which leads to the Itô formula (3.33). This in turn allows us to find the semimartingale description of the environment process and the respective Itô formula, see Section 4. Using these tools we present the proofs of our main results in Sections 5 – 7.

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2 Main results

2.1 Gaussian incompressible vector fields

Let us first make precise our assumptions on the random field $V(t,x)=(V_1,\dots,V_d)$. It is a mean-zero, space-time stationary d-dimensional Gaussian random field, defined on a probability space $(\Omega,\mathcal{F},\mathbb{P})$, with a covariance matrix of the form

$$R_{ij}(t,x)=\mathbb{E}\left[V_i(s+t,y+x)V_j(s,y)\right]=\int_{\mathbb{R}^d} e^{ix\cdot k}e^{-\alpha(k)|t|}\Gamma_{ij}(k)\sigma(k)dk, \quad i,j=1,\dots d. \tag{2.1}$$

The factor

$$\Gamma(k):=[\Gamma_{ij}(k)], \quad \Gamma_{ij}(k):=\delta_{i,j}-k_ik_j/|k|^2, \quad i,j=1,\dots,d,$$

ensures that the realizations of the field are almost surely incompressible:

$$\nabla_x \cdot V(t,x) = \sum_{j=1}^{d} \partial_{x_j} V_j(t,x) \equiv 0, \quad (t,x) \in \mathbb{R}^{1+d}, \text{ a.s.}$$

The temporal factor taking the form of $e^{-\alpha(k)|t|}$ plays an important role for our construction of the underlying Markovian dynamics. The non-negative functions $\alpha(k) \geq 0$ and $\sigma(k) \geq 0$ are assumed to be even and continuous. We also assume that $\sigma(k)$ is compactly supported: $\sigma(k) = 0$ for $|k| \geq K_0$, and the spectral gap $\alpha(k)$ is uniformly positive:

$$0 < \alpha_* \leq \alpha(k) \leq A_*, \quad k \in \mathbb{R}^d. \tag{2.2}$$

This property implies uniform mixing in time of the velocity field, see (3.15) and (3.16) below.

In order to specify the function space where $V(t,x)$ takes its values, given $m_1, m_2 \in \mathbb{R}$, let $\mathcal{E}_{m_1,m_2}$ be the real Hilbert space of vector-valued functions $w : \mathbb{R}^d \to \mathbb{R}^d$ with the norm

$$\|w\|_{m_1,m_2}^2 := \int_{\mathbb{R}^d} \theta_{-m_2}(x)[\mathcal{F}^{-1}(\theta_{m_1}\hat{w})(x)]^2 dx, \quad \theta_m(x) := (1+|x|^2)^{m/2}.$$

Here, the Fourier transform and its inverse are defined as

$$\hat{w}(k) = [\mathcal{F}w](k) := \int_{\mathbb{R}^d} e^{-ik\cdot x} w(x)dx, \quad [\mathcal{F}^{-1}u](x) := \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{ik\cdot x}\hat{u}(k)dk, \quad w,u \in \mathcal{S}(\mathbb{R}^d).$$

It is straightforward to check that the dual space $\mathcal{E}'_{m_1,m_2}$ to $\mathcal{E}_{m_1,m_2}$ is $\mathcal{E}_{-m_1,-m_2}$. Note that the Dirac function $\delta(x)$ belongs to $\mathcal{E}'_{m_1,m_2}$, provided that $m_1 > d$ and $m_2 \in \mathbb{R}$.

Under the above assumptions, for a fixed $t \in \mathbb{R}$, the realizations of the components of $V(t,\cdot)$ belong a.s. to any $\mathcal{E}_{m_1,m_2}$, with $m_1 \in \mathbb{R}$ and $m_2 > d$. Let $\mathcal{E}$ be the Hilbert space consisting of vector fields $w = (w_1,\ldots,w_d) : \mathbb{R}^d \to \mathbb{R}^d$ whose components belong to $\mathcal{E}_{m_1,m_2}$ for some $m_1 \geq 1$ and $m_2 > d$ satisfying $\nabla_x \cdot w(x) \equiv 0$, and let $\mathcal{B}(\mathcal{E})$ be its Borel σ-algebra. We denote by π the law of $V(0,\cdot)$ (which coincides with the law of $V(t,\cdot)$ for any $t \in \mathbb{R}$, due to stationarity) over the space $(\mathcal{E},\mathcal{B}(\mathcal{E}))$.

2.2 The linear case

Let us first consider

$$\partial_t u_\varepsilon(t,x) + \frac{1}{\varepsilon} V\left(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}\right) \cdot \nabla_x u_\varepsilon(t,x) = 0, \quad u_\varepsilon(T,x) = u_0(x), \quad 0 \leq t \leq T, \tag{2.3}$$

with a terminal condition u_0 that belongs to $C_0^\infty(\mathbb{R}^d)$. The solution of (2.3) is

$$u_\varepsilon(t,x) = u_0\big(X_\varepsilon^{t,x}(T)\big), \tag{2.4}$$

where we recall that $X_\varepsilon^{t,x}(s)$ is the characteristic curve defined in (1.3)

$$\begin{aligned} &\frac{dX_\varepsilon^{t,x}(s)}{ds} = \frac{1}{\varepsilon} V\Big(\frac{s}{\varepsilon^2}, \frac{X_\varepsilon^{t,x}(s)}{\varepsilon}\Big), \quad s > t, \\ &X_\varepsilon^{t,x}(t) = x. \end{aligned} \tag{2.5}$$

It is well known, see [7, 11, 20], that under our assumptions on $V(t,x)$, the laws of $\big(X_\varepsilon^{t,x}(s)\big)_{s\geq t}$ converge, as $\varepsilon \to 0$, to the law of $(x+\beta_{s-t})_{s\geq t}$. Here, $\beta_t = (\beta_t^1,\ldots,\beta_t^d)$, is a d-dimensional Brownian motion with the covariance

$$\mathbb{E}[\beta_t^p \beta_s^q] = a_{pq}(t \wedge s), \quad p,q = 1,\ldots,d,\ t,s \geq 0, \tag{2.6}$$

and the effective diffusivity matrix a_{pq} given by (4.10) below.

The above implies, in particular, that for each (t,x) fixed, $t \le T$, the random variables $u_\varepsilon(t,x)$, converge in law to a random variable $u_0(x+\beta_{T-t})$. In addition, $\bar{u}(t,x) := \mathbb{E}\left[u_0\left(x+\beta_{T-t}\right)\right]$ is the bounded solution of the backward heat equation

$$\begin{aligned}&\partial_t \bar{u}(t,x) + \frac{1}{2}\sum_{p,q=1}^{d} a_{pq}\partial^2_{x_p,x_q}\bar{u}(t,x) = 0, \quad t \le T,\\ &\bar{u}(T,x) = u_0(x).\end{aligned} \tag{2.7}$$

For the multi-point statistics we have the following.

Theorem 2.1. *For a given positive integer N, mutually distinct points $x_1,\dots,x_N \in \mathbb{R}^d$, and a fixed $t \le T$, the random vectors $(u_\varepsilon(t,x_1),\dots,u_\varepsilon(t,x_N))$ converge in law, as $\varepsilon \to 0$, to*

$$\left(u_0\left(x_1+\beta^{(1)}_{T-t}\right),\dots,u_0\left(x_1+\beta^{(N)}_{T-t}\right)\right),$$

where $(\beta^{(j)}_t)_{t\ge 0}$, $j=1,\dots,N$ are i.i.d. d-dimensional Brownian motions, each one with the covariance matrix given by (2.6).

Our result in the linear case allows, in particular, to contrast the point-wise convergence of $u_\varepsilon(t,x)$ to a random limit, with the convergence of $u_\varepsilon(t,\cdot)$ in the weak topology in $L^2(\mathbb{R}^d)$ to a deterministic limit. We use the notation

$$\langle f,g\rangle := \int_{\mathbb{R}^d} f(x)g(x)dx$$

for $f \in L^p(\mathbb{R}^d)$, $g \in L^{p'}(\mathbb{R}^d)$, with $1/p + 1/p' = 1$, $p \in [1,+\infty]$.

Corollary 2.2. *For a given $\varphi \in L^1(\mathbb{R}^d)$ and $t \le T$, the random variables*

$$\lim_{\varepsilon\to 0}\langle u_\varepsilon(t),\varphi\rangle = \langle \bar{u}(t),\varphi\rangle \quad \textit{in the } L^2(\Omega) \textit{ sense.}$$

The proofs of the above results are presented in Section 5.

Remark. The reason why we obtain a deterministic limit in Corollary 2.2 is due to the spatial averaging, which removes the local fluctuations after testing with φ. If one is interested in the local fluctuation, i.e., the fluctuations averaged on a “small scale” $\delta^{-d}\int_{\mathbb{R}^d}\varphi(x/\delta)u_\varepsilon(t,x)dx$ with $\delta \ll 1$, then the pointwise quantity $u_\varepsilon(t,x)$ might be a more relevant object ($\delta \to 0$) compared to the global fluctuation described by $\int u_\varepsilon(t,x)\varphi(x)dx$ ($\delta = 1$).

Let us also comment that when $u_\varepsilon(t,x)$ satisfies an advection-diffusion equation rather than an advection equation, as in (2.3),

$$\begin{aligned}&\partial_t u_\varepsilon(t,x) + \frac{1}{\varepsilon}V\left(\frac{t}{\varepsilon^2},\frac{x}{\varepsilon}\right)\cdot\nabla_x u_\varepsilon(t,x) + \kappa\Delta_x u_\varepsilon(t,x) = 0,\\ &u_\varepsilon(T,x) = u_0(x),\end{aligned} \tag{2.8}$$

with $\kappa > 0$, one can prove, see [15], that for any $t \le T$ both $u_\varepsilon(t,x)$ and $\langle u_\varepsilon(t),\varphi\rangle$ converge in probability to deterministic limits $\bar{u}(t,x)$ and $\langle\bar{u}(t),\varphi\rangle$, respectively. In that case, $\bar{u}(t,x)$ is the solution of the Cauchy problem for the backward heat equation

$$\begin{aligned}&\partial_t \bar{u}(t,x) + \frac{1}{2}\sum_{p,q=1}^{d} a_{pq}\partial^2_{x_p,x_q}\bar{u}(t,x) + \kappa\Delta_x\bar{u}(t,x) = 0, \quad t \le T,\\ &\bar{u}(T,x) = u_0(x).\end{aligned} \tag{2.9}$$

In other words, the diffusion term in (2.8) provides enough extra averaging so that even the point-wise limit is deterministic.

2.3 The semi-linear case

Let $D_T := [0,T]\times\mathbb{R}^{d+1}$ and $C^{0,m}(D_T)$ be the space of continuous functions $g(t,x,u)$ on D_T, that are of the class C^m in the (x,u) variables for some non-negative integer m:

$$\|g\|_{C^{0,m}(D_T)} := \sum_{|k|=0}^{m} \sup_{(t,x,u)\in D_T} |D^k_{x,u} g(t,x,u)|.$$

We consider semi-linear equations of the form

$$\begin{aligned} &\partial_t u_\varepsilon(t,x) + \frac{1}{\varepsilon} V\Big(\frac{t}{\varepsilon^2},\frac{x}{\varepsilon}\Big)\cdot\nabla_x u_\varepsilon(t,x) = f_\varepsilon(t,x,u_\varepsilon(t,x)), \quad t<T,\\ &u_\varepsilon(T,x) = u_0(x), \end{aligned} \tag{2.10}$$

with

$$f_\varepsilon(t,x,u) := f\Big(t,x,u,V\Big(\frac{t}{\varepsilon^2},\frac{x}{\varepsilon}+\cdot\Big)\Big), \tag{2.11}$$

and $f(\cdot,w)\in C^{0,m}(D_T)$ for some $m>(d+1)/2$ and π-a.s. $w\in\mathcal{E}$, and

$$\operatorname*{esssup}_{w\in\mathcal{E}} \|f(\cdot,w)\|_{C^{0,m}(D_T)} < +\infty. \tag{2.12}$$

Note that we omitted the dependence of f_ε on the random realization ω to simplify the notation. As $f_\varepsilon(t,x,u)$ is now random, the results will depend on whether it has a zero or non-zero mean

$$\bar f(t,x,u) := \mathbb{E} f\left(t,x,u,V(0,\cdot)\right), \quad (t,x,u)\in D_T, \tag{2.13}$$

and we will consider these two cases separately. In the non-centered case we have the following result, proved in Section 6.

Theorem 2.3. *Assume that $\bar f(t,x,u)\not\equiv 0$ in D_T. Fix $(t,x)\in[0,T]\times\mathbb{R}^d$ and the realization of the Brownian motion β_t with the covariance matrix (2.6), and let $\{\mathcal{U}(s;t,x)\}_{t\le s\le T}$ satisfy the integral equation*

$$u_0\left(x+\beta_{T-t}\right) - \mathcal{U}(s;t,x) = \int_s^T \bar f\left(\sigma, x+\beta_{\sigma-t}, \mathcal{U}(\sigma;t,x)\right) d\sigma, \quad t\le s\le T. \tag{2.14}$$

Then $u_\varepsilon(t,x)$ converges in law, as $\varepsilon\to 0$, to $\mathcal{U}(t,x) := \mathcal{U}(t;t,x)$. Moreover, for any positive integer N, mutually distinct $x_1,\dots,x_N\in\mathbb{R}^d$ and $t\le T$ the random vectors $(u_\varepsilon(t,x_1),\dots,u_\varepsilon(t,x_N))$ converge in law, as $\varepsilon\to 0$, to $(\mathcal{U}^{(1)}(t,x_1),\dots,\mathcal{U}^{(N)}(t,x_N))$, where $\mathcal{U}^{(1)},\dots,\mathcal{U}^{(N)}$ correspond to the solutions of (2.14) with β replaced by i.i.d. copies of d-dimensional Brownian motions $(\beta^{(j)}_t)_{t\ge 0}$, $j=1,\dots,N$, whose covariance matrix is given by (2.6).

From the above result, we conclude an analogue of Corollary 2.2:

Corollary 2.4. *Suppose that $\varphi\in L^1(\mathbb{R}^d)$. Then*

$$\lim_{\varepsilon\to 0}\langle u_\varepsilon(t),\varphi\rangle = \langle \mathbb{E}\,\mathcal{U}(t),\varphi\rangle \quad \textit{in the } L^2(\Omega) \textit{ sense.} \tag{2.15}$$

If $\bar f(t,x,u)\equiv 0$ for $(t,x,u)\in D_T$, the leading order effect of f in (2.14) vanishes, so, to have the nonlinearity play a non-trivial role, we consider instead of (2.10) the problem

$$\begin{aligned} &\partial_t u_\varepsilon(t,x) + \frac{1}{\varepsilon} V\left(\frac{t}{\varepsilon^2},\frac{x}{\varepsilon}\right)\cdot\nabla_x u_\varepsilon(t,x) = \frac{1}{\varepsilon} f_\varepsilon(t,x,u_\varepsilon(t,x)), \quad t<T,\\ &u_\varepsilon(T,x) = u_0(x). \end{aligned} \tag{2.16}$$

Here, f_ε is as in (2.11). We will, however, require slightly more regularity on f_ε. Let $C^m(D_T)$ be the space of continuous functions $g : D_T \times \mathcal{E} \to \mathbb{R}$, that are of the class C^m in the (s, x, u) variables for some non-negative integer m:

$$\|g\|_{C^m(D_T)} := \sum_{|k|=0}^{m} \sup_{(s,y,u)\in D_T} |D^k_{s,y,u} g(s, y, u)|.$$

We assume that $f(t, x, u, \omega)$ is such that $f(\cdot, w) \in C^m(D_T)$ for some $m > (d+1)/2$ and π-a.s. $w \in \mathcal{E}$, and

$$\operatorname*{esssup}_{w\in\mathcal{E}} \|f(\cdot, w)\|_{C^m(D_T)} < +\infty. \tag{2.17}$$

In order to state the result, let $U^{t,x,u}_\varepsilon(s)$ be the solution of (2.16) along the characteristics $X^{t,x}_\varepsilon(s)$ given by (2.5), satisfying $U^{t,x,u}_\varepsilon(t) = u$. In other words, it is the solution of the equation

$$U^{t,x,u}_\varepsilon(s) = u + \frac{1}{\varepsilon}\int_t^s f_\varepsilon(\sigma)d\sigma, \quad t \le s \le T, \tag{2.18}$$

where for $f : \mathbb{R}^{d+2} \times \mathcal{E} \to \mathbb{R}$ we simply write

$$f_\varepsilon(\sigma) := f\Big(\sigma, X^{t,x}_\varepsilon(\sigma), U^{t,x,u}_\varepsilon(\sigma), V\Big(\frac{\sigma}{\varepsilon^2}, \frac{X^{t,x}_\varepsilon(\sigma)}{\varepsilon} + \cdot\Big)\Big).$$

For a fixed pair (t, x), we define a random field

$$\mathfrak{S}^{t,x}_\varepsilon(s, u) := U^{t,x,u}_\varepsilon(s), \quad t \le s \le T, \quad u \in \mathbb{R}. \tag{2.19}$$

In order to define the limit of $\mathfrak{S}^{t,x}_\varepsilon(s, u)$, let us introduce the solution of the following system of Itô stochastic differential equations

$$\begin{aligned} U^{t,x,u}(s) &= u + \int_t^s b(\sigma, X^{t,x}(\sigma), U^{t,x,u}(\sigma))d\sigma + \sum_{j=0}^{d}\int_t^s \tilde{c}_j(\sigma, X^{t,x}(\sigma), U^{t,x,u}(\sigma))d\tilde{\beta}_j(\sigma), \\ X^{t,x}_j(s) &= x_j + \sum_{k=1}^{d}\int_t^s S_{jk}d\tilde{\beta}_k(\sigma), \quad j = 1, \dots, d. \end{aligned} \tag{2.20}$$

Here, the coefficients $b(s, x, u)$ and $\tilde{c}_j(s, x, u)$ are defined in Section 7.1 below, $\tilde{\beta}_j(\sigma)$, $j = 0, \dots, d$, are i.i.d. standard Brownian motions, and S_{jk} is the square root of the $d \times d$ matrix a_{jk}, given by (2.6). The limiting dynamics (2.20) has the following property proved in Section 7.1.

Proposition 2.5. *Given (t, x), and $s \in [t, T]$, let $\mathfrak{s}^{t,x}_s(u) := U^{t,x,u}(s)$. The mapping $\mathfrak{s}^{t,x}_s : \mathbb{R} \to \mathbb{R}$ is a.s. a diffeomorphism.*

This leads to the main result concerning the convergence of the solution of (2.16).

Theorem 2.6. *(i) The joint laws of $\big(X^{t,x}_\varepsilon(\cdot), \mathfrak{S}^{t,x}_\varepsilon(\cdot)\big)$, over $C([t, T]) \times C([t, T] \times \mathbb{R})$, equipped with the standard Frechet metric metrizing uniform convergence on compact sets, converge weakly to the law of $\big(X^{t,x}(\cdot), \mathfrak{S}^{t,x}(\cdot)\big)$, with $\mathfrak{S}^{t,x}(s, u) := U^{t,x,u}(s)$.*
(ii) For each $(t, x) \in \mathbb{R}^{1+d}$ fixed, the random variables $u_\varepsilon(t, x)$ converge in law, as $\varepsilon \to 0$, to

$$\mathscr{U}(t, x) := (\mathfrak{s}^{t,x}_T)^{-1}(u_0(X^{t,x}(T))). \tag{2.21}$$

In addition, for any positive integer N, mutually distinct $x_1, \dots, x_N \in \mathbb{R}^d$ and $t \le T$, the random vectors $(u_\varepsilon(t, x_1), \dots, u_\varepsilon(t, x_N))$ converge in law, as $\varepsilon \to 0$, to $(\mathscr{U}^{(1)}(t, x_1), \dots, \mathscr{U}^{(N)}(t, x_N))$, where $\mathscr{U}^{(1)}, \dots, \mathscr{U}^{(N)}$ correspond, via (2.21), to $(\mathfrak{s}^{t,x_j}_T(\cdot), X^{t,x_j}_j)$, $j = 1, \dots, N$ driven by i.i.d. copies of d-dimensional standard Brownian motions as in (2.20).

3 Some preliminaries on Gaussian, Markovian fields

In this section, we give a Markovian representation for the field $V(t,x)$, starting from the assumptions in Section 2.1. To this end, let $\mathcal{H}_1$ be the L^2-closure of the linear space spanned by the random variables

$$W(\varphi;w)=\sum_{j=1}^d\int_{\mathbb{R}^d}w_j(x)\varphi_j(x)dx,\quad \varphi\in\mathcal{S}_{\rm div}(\mathbb{R}^d;\mathbb{R}^d),\quad w\in\mathcal{E},\tag{3.1}$$

defined over the probability space $(\mathcal{E},\mathcal{B}(\mathcal{E}),\pi)$. Here, $\mathcal{S}_{\rm div}(\mathbb{R}^d;\mathbb{R}^d)$ is the space of divergence free vector fields $\varphi=(\varphi_1,\dots,\varphi_d)$ with components in $\mathcal{S}(\mathbb{R}^d)$. By an approximation argument, W extends to a unitary mapping $W:H\to\mathcal{H}_1$, where H is the (real) Hilbert space, the closure of $\mathcal{S}_{\rm div}(\mathbb{R}^d;\mathbb{R}^d)$ in the norm $\|\cdot\|_H$, with

$$\langle\varphi_1,\varphi_2\rangle_H:=\int_{\mathbb{R}^d}\hat\varphi_1(k)\cdot\hat\varphi_2^*(k)\sigma(k)dk,\quad \varphi_1,\varphi_2\in\mathcal{S}_{\rm div}(\mathbb{R}^d;\mathbb{R}^d).\tag{3.2}$$

Here, $\sigma(k)$ is as in (2.1). In addition, by (2.1) and the fact that φ_1,φ_2 are divergence free, we have

$$\langle W(\varphi_1;w),W(\varphi_2;w)\rangle_{L^2(\pi)}=\langle\varphi_1,\varphi_2\rangle_H,\quad \varphi_1,\varphi_2\in H.\tag{3.3}$$

Note that the shift

$$\tau_x\varphi(\cdot):=\varphi(x+\cdot)$$

is an isometry on H, for each $x\in\mathbb{R}^d$. In the following, we will simply write

$$W(\varphi)=W(\varphi;w).$$

3.1 The Gaussian chaos expansion

Let $\mathcal{P}_n$ be the space of the n-th degree polynomials, the L^2-closure of the linear span of

$$\prod_{j=1}^m W(\varphi_j),\quad 1\le m\le n,\quad \varphi_1,\dots,\varphi_n\in H,$$

and $\mathcal{H}_n:=\mathcal{P}_n\ominus\mathcal{P}_{n-1}$, $n\ge1$ be the space of the n-th degree Hermite polynomials, with the convention $\mathcal{H}_0=\mathcal{P}_0=\mathbb{R}$. It is well known, see e.g. Theorem 2.6, p. 18 of [14], that

$$L^2(\pi)=\bigoplus_{n=0}^{+\infty}\mathcal{H}_n.$$

Denote by $\mathfrak{p}_n$ the orthogonal projection of $L^2(\pi)$ onto $\mathcal{H}_n$. Given $s\in[0,+\infty)$, the Hilbert space $\mathfrak{H}_s$ is made of $F\in L^2(\pi)$ with $\mathfrak{p}_0F=0$ and the norm

$$\|F\|_{\mathfrak{H}_s}:=\Big\{\sum_{n=1}^{+\infty}(1+n)^s\|\mathfrak{p}_nF\|^2_{L^2(\pi)}\Big\}^{1/2}<+\infty.\tag{3.4}$$

We set

$$\mathfrak{H}_\infty:=\bigcap_{s\ge0}\mathfrak{H}_s.\tag{3.5}$$

The homogeneity assumption on π amounts to the fact that $\pi\tau_x=\pi$ for each $x\in\mathbb{R}^d$. Therefore, the operators $T_xF(w):=F(\tau_xw)$, $w\in\mathcal{E}$, $x\in\mathbb{R}^d$, form a strongly continuous group of isometries on $L^p(\pi)$ for any $p \in [1, +\infty)$. Denote by $D = (D_1, \ldots, D_d)$ the generators of $(T_x)_{x\in\mathbb{R}^d}$. Let $\mathcal{W}_{k,p}$ be the Banach space consisting of $F \in L^p(\pi)$ that belong to the domain of $D^m = \prod_{j=1}^d D_j^{m_j}$, for a non-negative integer multi-index $m = (m_1, \ldots, m_d)$, with $|m| := \sum_{i=1}^d m_i \le k$, equipped with the norm

$$\|F\|_{k,p} := \Big\{ \sum_{|m|\le k} \|D^m F\|^p_{L^p(\pi)} \Big\}^{1/p}. \tag{3.6}$$

The space $\mathcal{W}_{k,\infty}$ is defined with the help of L^∞ norm.

We let $\mathcal{W}_\infty := \bigcap_{p>1} \mathcal{W}_{1,p}$. It follows from the definition of T_x that

$$T_x\Big(\prod_{j=1}^n W(\varphi_j) \Big) = \prod_{j=1}^n W(\tau_{-x}\varphi_j), \quad \varphi_1, \ldots, \varphi_n \in H.$$

Therefore, $T_x(\mathcal{P}_n) = \mathcal{P}_n$, and since T_x is unitary on $L^2(\pi)$, we also get $T_x(\mathcal{H}_n) = \mathcal{H}_n$ for all $n \ge 0$. Due to the assumption that σ is compactly supported, we conclude easily that $\mathcal{P} := \bigcup_{n\ge 0} \mathcal{P}_n \subset \mathcal{W}_\infty$.

Finally, we define the linear functionals $v_p : \mathcal{E} \to \mathbb{R}$ as

$$v_p(w) := w_p(0), \quad w \in \mathcal{E},\ p = 1, \ldots, d. \tag{3.7}$$

They are bounded and can be written as

$$v_p = W(f_p), \quad p = 1, \ldots, d, \tag{3.8}$$

with $f_p \in H$ given by

$$f_p(x) := \int_{\mathbb{R}^d} e^{ik\cdot x}\Gamma(k)\mathrm{e}_p dk, \quad \mathrm{e}_p = \underbrace{(0, \ldots, 1, \ldots, 0)}_{p\text{-th position}}.$$

3.2 Markovian dynamics of the velocity field

Here, we formulate the Markov property of the $\mathcal{E}$-valued process $V_t := V(t, \cdot)$, $t \in \mathbb{R}$. We represent the random field $V(t, x)$ in the form

$$V(t, x) = v(\tau_x V_t), \quad (t, x) \in \mathbb{R}^{1+d}, \tag{3.9}$$

with $v = (v_1, \ldots, v_d)$ as in (3.7). Given $t \ge 0$, let $S_t : H \to H$ be the continuous extension of

$$\widehat{S_t\varphi}(k) := e^{-\alpha(k)t}\widehat{\varphi}(k), \quad \varphi \in \mathcal{S}_{\mathrm{div}}(\mathbb{R}^d; \mathbb{R}^d). \tag{3.10}$$

The family $(S_t)_{t\ge 0}$ is a C_0-semigroup of symmetric contractions on H, with the generator $(-A)$ and

$$\widehat{A\varphi}(k) := \alpha(k)\hat{\varphi}(k), \quad \varphi \in \mathcal{S}_{\mathrm{div}}(\mathbb{R}^d; \mathbb{R}^d). \tag{3.11}$$

One can easily verify that P_t, defined via

$$P_t W(\varphi) := W(S_t\varphi), \quad t \ge 0, \quad \varphi \in H, \tag{3.12}$$

forms a semigroup of contractions on $\mathcal{H}_1$. Similarly, for $F = \mathfrak{p}_n\left(\prod_{j=1}^n W(\varphi_j)\right)$, we set

$$P_t F := \mathfrak{p}_n \left(\prod_{j=1}^n W(S_t\varphi_j) \right). \tag{3.13}$$

According to Theorem 4.5 of [14], $(P_t)_{t\geq 0}$ is a contraction semigroup on $\mathcal{H}_n$ for each n, hence a semigroup of contractions on the entire $L^2(\pi)$. It can be easily checked (using e.g. Theorem 3.9, p. 26 of [14]) that P_t forms a strongly continuous semigroup of symmetric operators on $L^2(\pi)$.

Let $\mathfrak{V}_s$ be the L^2 closure of the linear span of $W(\varphi; V_u)$ for any $u \leq s$ and $\varphi \in H$. For any $t \geq s$ and $\varphi \in H$, the orthogonal projection of $W(\varphi; V_t)$ onto $\mathfrak{V}_s$ is

$$W(S_{t-s}\varphi; V_s) = P_{t-s}W(\varphi)(V_s).$$

Therefore, according to Theorem 4.9, p. 46 of [14], for any $F \in L^2(\pi)$ and $t \geq s$ we have

$$\mathbb{E}\left[F(V_t) \mid \mathcal{V}_s\right] = P_{t-s}F(V_s),$$

where $(\mathcal{V}_s)$ is the natural filtration of $(V_t)_{t\geq 0}$. Note that for any $x \in \mathbb{R}^d$, $t \geq 0$ we have

$$S_t(\tau_x(\varphi)) = \tau_x(S_t\varphi).$$

Using (3.13), we can conclude also that

$$P_tT_x = T_xP_t, \quad t \geq 0,\, x \in \mathbb{R}^d. \tag{3.14}$$

It follows from (3.10) and (3.12) that

$$\|P_tF\|_{L^2(\pi)} \leq e^{-\alpha_* t}\|F\|_{L^2(\pi)}, \quad t \geq 0, \quad F \in \mathcal{H}_1. \tag{3.15}$$

Using Theorem 4.5, p. 46 of [14] we conclude that (3.15) actually holds for any $F \in L^2(\pi)$ such that $\langle F, 1\rangle_{L^2(\pi)} = 0$. We denote by $L : \mathcal{D}(L) \to L^2(\pi)$ the L^2-generator of P_t, which, due to the symmetry of the semigroup, is self-adoint. Since $\mathcal{P}$ is dense in $L^2(\pi)$ and invariant under $(P_t)_{t\geq 0}$, it is a core of L, see e.g. Proposition 3.3, p. 17 of [10]. As a consequence of (3.15) we have an estimate for the Dirichlet form

$$\mathcal{E}_L(F) := -\langle LF, F\rangle_{L^2(\pi)} = -\lim_{t\to 0}\frac{1}{t}\left(\langle P_tF, F\rangle - \langle F, F\rangle\right) \geq \alpha_*\|F\|^2_{L^2(\pi)}, \quad F \in \mathcal{D}(L),\, F \perp 1. \tag{3.16}$$

In fact, we have an estimate that allows us to compare the Dirichlet form with the L^2 and $\|\cdot\|_{1,2}$ norms on the space of the n-th degree Hermite polynomials.

Theorem 3.1. *The following estimates hold:*

(i)

$$\alpha_* n\|F\|^2_{L^2(\pi)} \leq \mathcal{E}_L(F) \leq A_* n\|F\|^2_{L^2(\pi)}, \quad F \in \mathcal{H}_n, \quad n \geq 0. \tag{3.17}$$

(ii) There exists a constant $C > 0$ such that

$$\sum_{j=1}^{d}\|D_jF\|^2_{L^2(\pi)} \leq Cn\mathcal{E}_L(F), \quad F \in \mathcal{H}_n, \quad n \geq 0. \tag{3.18}$$

(iii) There exists a constant $C > 0$ such that

$$\|v_pF\|_{L^2(\pi)} \leq C\mathcal{E}_L^{1/2}(F), \quad F \in \mathcal{H}_n, \quad n \geq 1, \quad p = 1, \ldots, d. \tag{3.19}$$

The proof of part (i) is presented in Section 3.4. The proofs of parts (ii) and (iii) can be found in [16], see the estimate (12.115), p. 413 and Lemma 12.25, p. 405, respectively. As a direct conclusion from the above result, we obtain the following (cf (3.4)).

Corollary 3.2. *We have $\mathcal{D}(\mathcal{E}_L) = \mathfrak{H}_1$.*

3.3 A stochastic convolution representation for the velocity field

In order to obtain a more explicit representation for V_t, note that given $\varphi \in H$, the process $V_t(\varphi) := W(\varphi; V_t)$ is a Gaussian semimartingale satisfying

$$dV_t(\varphi) = -V_t(A\varphi)dt + \sqrt{2}dB_t(\varphi), \quad t \geq s, \, \varphi \in H, \tag{3.20}$$

for any $s \in \mathbb{R}$. Here, the process $B : \mathbb{R} \times H \times \Omega \to \mathbb{R}$ is such that the process $((B_t(\varphi_1), \ldots, B_t(\varphi_n))_{t \in \mathbb{R}}$ is an n-dimensional, two sided, Brownian motion, with zero mean and covariance

$$\mathbb{E}[B_t(\varphi_i)B_s(\varphi_j)] = (t \wedge s)\langle A\varphi_i, \varphi_j \rangle_H, \quad i, j = 1, \ldots, n, \quad t, s \in \mathbb{R}, \tag{3.21}$$

for any $\varphi_1, \ldots, \varphi_n \in H$. In addition, for any $s \in \mathbb{R}$ the process $(B_t - B_s)_{t \geq s}$ is independent of $\mathcal{V}_s$ - the σ-algebra generated by V_u, $u \leq s$.

Suppose that an H-valued process $(\varphi_t)_{t \geq s}$ is progressively measurable w.r.t. the filtration $\mathcal{V}_t$ and satisfies

$$\int_s^t \mathbb{E}\|\varphi_u\|_H^2 du < +\infty.$$

By the standard procedure, we can define the Itô integral

$$\int_s^t dB_u(\varphi_u),$$

sometimes also denoted by

$$\int_s^t \langle \varphi_u, dB_u \rangle_H.$$

It is a square integrable, zero mean, continuous trajectory martingale that satisfies

$$\mathbb{E}\left[\int_s^t dB_u(\varphi_u)\right]^2 = \int_s^t \mathbb{E}\langle A\varphi_u, \varphi_u \rangle_H du.$$

Stationary, Gaussian and Markovian process (V_t) can be represented as a stochastic convolution.

Proposition 3.3. *For any $\varphi \in H$ we can write*

$$V_t(\varphi) = \sqrt{2}\int_{-\infty}^t \langle S_{t-s}\varphi, dB_s \rangle_H, \quad t \in \mathbb{R}. \tag{3.22}$$

3.4 Proof of (3.17)

Recall that the n-th degree, L^2-normalized, Hermite polynomial $h_n(x)$ is

$$h_0(x) \equiv 1, \qquad h_n(x) := \frac{(-1)^n}{\sqrt{n!}} e^{x^2/2} \frac{d^n}{dx^n}\left(e^{-x^2/2}\right), \quad x \in \mathbb{R}.$$

It is well known that

$$xh_n(x) = (n+1)^{1/2}h_{n+1}(x) + n^{1/2}h_{n-1}(x), \quad h_n'(x) = \sqrt{n}h_{n-1}(x), \quad n \geq 1. \tag{3.23}$$

Suppose that $(\mathfrak{e}_j)_{j \geq 1}$ is an orthonormal base in H. Let $\mathbf{n} = (n_j)_{j \geq 1}$ be a sequence of non-negative integers, and $|\mathbf{n}| := \sum_{j=1}^{+\infty} n_j$. According to Proposition 1.1.1 of [26], the vectors

$$h_{\mathbf{n}} := \prod_{j=1}^{+\infty} h_{n_j}\left(W(\mathfrak{e}_j)\right), \quad |\mathbf{n}| = n,$$

form an orthonormal base in $\mathcal{H}_n$. Suppose that $F=\sum_{n=0}^{+\infty}\mathfrak{p}_nF\in L^2(\pi)$ and $\mathfrak{p}_n(F)=\sum_{|\mathbf{n}|=n}\alpha_{\mathbf{n}}h_{\mathbf{n}}$ for some real coefficients $(\alpha_{\mathbf{n}})$ satisfying

$$\sum_{n=0}^{+\infty}\sum_{|\mathbf{n}|=n}\alpha_{\mathbf{n}}^2=\|F\|^2_{L^2(\pi)}<+\infty.$$

The operator $\mathcal{D}:\mathfrak{H}_1\to L^2(\pi;H)$ defined as

$$\mathcal{D}F:=\sum_{j=1}^{+\infty}\mathcal{D}_jF\mathfrak{e}_j \tag{3.24}$$

with $\mathcal{D}_j:\mathfrak{H}_1\to L^2(\pi)$, $j=1,2,\ldots$ given by

$$\mathcal{D}_jF:=\sum_{n=0}^{+\infty}\sum_{|\mathbf{n}|=n}\sqrt{n_j}\alpha_{\mathbf{n}}h_{\mathbf{n}_j},\quad F\in\mathfrak{H}_1, \tag{3.25}$$

is the Malliavin derivative, see Definition 1.2.1, p. 25 of [26]. Here, $\mathbf{n}_j=(n'_{j'})$ is given by

$$n'_{j'}:=\begin{cases}n_{j'}, & j'\neq j,\\ (n_{j'}-1)_+, & j'=j.\end{cases}$$

Note that, (cf Proposition 1.2.2, p. 28 of [26])

$$\sum_{j=1}^{+\infty}\|\mathcal{D}_jF\|^2_{L^2(\pi)}=\sum_{n=0}^{+\infty}n\sum_{|\mathbf{n}|=n}\alpha_{\mathbf{n}}^2=\sum_n n\|\mathfrak{p}_nF\|^2_{L^2(\pi)}<+\infty,\quad F\in\mathfrak{H}_1. \tag{3.26}$$

Remark. For F of the form $F:=\Phi(W(h_1),\ldots,W(h_N))$, where $h_1,\ldots,h_N\in H$ and $\Phi\in C^\infty(\mathbb{R}^N)$ with both Φ and its partial derivatives of polynomial growth, we have $\mathcal{D}F=\sum_{p=1}^N\partial_{x_p}\Phi h_p$.

Denote by $h_{\mathbf{n}}(t):=h_{\mathbf{n}}(V_t)$, and $B_j(t):=B_t(\mathfrak{e}_j)$, where B_t was defined in Section 3.3. Recall that (see (3.21))

$$\mathbb{E}[B_i(t)B_j(s)]=c_{i,j}(t\wedge s),\quad i,j=1,2,\ldots,$$

with $c_{i,j}$ given by

$$c_{i,j}:=\langle A\mathfrak{e}_i,\mathfrak{e}_j\rangle_H,\quad i,j=1,2,\ldots. \tag{3.27}$$

Using the Itô formula, (3.20) and (3.23), one can show by a direct calculation that

$$\begin{aligned}dh_{\mathbf{n}}(t)=\Big\{-\sum_j\sqrt{n_j}h_{\mathbf{n}_j}(t)V_t(A\mathfrak{e}_j)+\sum_{j_1\neq j_2}c_{j_1,j_2}\sqrt{n_{j_1}n_{j_2}}h_{\mathbf{n}_{j_1,j_2}}(t)\\ +\sum_j c_{j,j}\sqrt{n_j(n_j-1)_+}h_{\mathbf{n}'_j}(t)\Big\}dt+\sqrt{2}\sum_j\sqrt{n_j}h_{\mathbf{n}_j}(t)dB_j(t).\end{aligned} \tag{3.28}$$

Here, $\mathbf{n}_{j_1,j_2}=(m'_{j'})$, $\mathbf{n}'_j=(\ell'_{j'})$ are multi-indices given by

$$m'_{j'}:=\begin{cases}n_{j'}, & j'\notin\{j_1,j_2\},\\ (n_{j'}-1)_+, & j'\in\{j_1,j_2\},\end{cases}\quad \ell'_{j'}:=\begin{cases}n_{j'}, & j'\neq j,\\ (n_{j'}-2)_+, & j'=j.\end{cases}$$

For $F=\sum_n\sum_{|\mathbf{n}|=n}\alpha_{\mathbf{n}}h_{\mathbf{n}}$, by (3.23), (3.27) and (3.28), we have

$$\begin{aligned}\mathcal{E}_L(F)=&\sum_{n,j}\sum_{|\mathbf{n}|=|\mathbf{m}|=n}\sqrt{n_j}\alpha_{\mathbf{n}}\alpha_{\mathbf{m}}\langle W(A\mathfrak{e}_j)h_{\mathbf{n}_j},h_{\mathbf{m}}\rangle_{L^2(\pi)}\\=&\sum_{n,j,j'}\sum_{|\mathbf{n}|=|\mathbf{m}|=n}c_{j,j'}\sqrt{n_j}\alpha_{\mathbf{n}}\alpha_{\mathbf{m}}\langle h_{\mathbf{n}_j},W(\mathfrak{e}_{j'})h_{\mathbf{m}}\rangle_{L^2(\pi)}\\=&\sum_{n,j,j'}\sum_{|\mathbf{n}|=|\mathbf{m}|=n}c_{j,j'}\sqrt{n_jm_{j'}}\alpha_{\mathbf{n}}\alpha_{\mathbf{m}}\langle h_{\mathbf{n}_j},h_{\mathbf{m}_{j'}}\rangle_{L^2(\pi)}.\end{aligned}\tag{3.29}$$

Comparing with (3.24) and (3.25) we conclude the formula

$$\mathcal{E}_L(F)=\int_{\mathcal{E}}\langle A\mathcal{D}F,\mathcal{D}F\rangle_H d\pi,\quad F\in D(\mathcal{E}_L).\tag{3.30}$$

Thanks to the inequality

$$\alpha_*\|\varphi\|_H^2\le\langle A\varphi,\varphi\rangle_H\le A_*\|\varphi\|_H^2,\quad\forall\,\varphi\in H,\tag{3.31}$$

(following directly from (2.2)) and identity (3.26) we conclude that

$$\alpha_*\sum_n n\|\mathfrak{p}_nF\|^2_{L^2(\pi)}\le\mathcal{E}_L(F)\le A_*\sum_n n\|\mathfrak{p}_nF\|^2_{L^2(\pi)},\quad F\in\mathcal{D}(\mathcal{E}_L).\tag{3.32}$$

Hence F belongs to $\mathcal{D}(\mathcal{E}_L)$ – the domain of the form $\mathcal{E}_L(\cdot)$ iff $F\in\mathfrak{H}_1$, i.e.

$$\sum_n n\|\mathfrak{p}_nF\|^2_{L^2(\pi)}<+\infty.$$

Thus, in particular (3.17) follows. □

3.5 Some corollaries of Theorem 3.1

Note that $F\in\mathcal{D}(L)$ iff $G:=L^{1/2}F\in\mathcal{D}(\mathcal{E}_L)$, that is $G\in\mathfrak{H}_1$. However, according to part (i) of Theorem 3.1 then $\|\mathfrak{p}_nG\|^2_{L^2(\pi)}\asymp n\|\mathfrak{p}_nF\|^2_{L^2(\pi)}$. The symbol $a_n\asymp b_n$ used for two non-negative sequences $(a_n)_{n\ge1}$ and $(b_n)_{n\ge1}$ means that there exists $C>0$ such that $Ca_n\le b_n\le a_n/C$ for all n. We conclude the following.

Corollary 3.4. *We have $\mathcal{D}(L)=\mathfrak{H}_2$.*

Next, we write down an Itô formula for the process V_t that will also be of great use for us. From (3.28) we obtain that for any $F\in\mathcal{D}(L)$

$$F(V_t)=F(V_0)+\int_0^t LF(V_s)ds+\sqrt{2}M_t(F),\tag{3.33}$$

where $M_t(F)$ is a continuous, square integrable martingale given by

$$M_t(F)=\sum_{j=1}^{+\infty}\int_0^t\mathcal{D}_jF(V_s)dB_j(s),\tag{3.34}$$

where $\mathcal{D}F$ is the Malliavin derivative defined in (3.24).

4 The environment process and the corrector fields

Let $X^{t,x}(s)$ be the solution of (2.5) corresponding to $\varepsilon = 1$. The ($\mathcal{E}$-valued) *environment process* is

$$\eta_s^{t,x} := \tau_{X^{t,x}(s)} V_s, \quad s \geq t, \tag{4.1}$$

so that

$$X^{t,x}(s) = x + \int_t^s v(\eta_\sigma^{t,x}) d\sigma.$$

We shall write $X(s)$, η_s instead of $X^{0,0}(s)$ and $\eta_s^{0,0}$, respectively.

4.1 Properties of the environment process and the corrector

Let $\mathcal{V}_{t,s}$ be the σ-algebra generated by V_u, $t \leq u \leq s$ and $B_b(\mathcal{E})$ be the space of bounded Borel measurable functions $F : \mathcal{E} \to \mathbb{R}$. The following is a consequence of the results in [16, Section 12.10].

Proposition 4.1. *For a given (t, x), the natural filtration of $(\eta_s^{t,x})_{s \geq t}$ coincides with $(\mathcal{V}_{t,s})_{s \geq t}$. The process $(\eta_s^{t,x})_{s \geq t}$ is Markovian and stationary, that is, for any $s \geq t$ and $h \geq 0$ we have*

$$\mathbb{E}\left[F(\eta_{s+h}^{t,x}) \mid \mathcal{V}_{t,s}\right] = Q_h F(\eta_s^{t,x}), \qquad a.s., \text{ where } F \in B_b(\mathcal{E}),$$

and

$$Q_s F(w) = \mathbb{E}\left[F(\eta_s) \mid \eta_0 = w\right], \quad s \geq 0, \ \pi \text{ a.s. in } w \in \mathcal{E}, \text{ where } F \in L^2(\pi).$$

In addition, π is invariant under $(Q_s)_{s \geq 0}$:

$$\int_{\mathcal{E}} Q_s F d\pi = \int_{\mathcal{E}} F d\pi, \quad s \geq 0, \ F \in B_b(\mathcal{E}), \tag{4.2}$$

and $(Q_s)_{s \geq 0}$ extends to C_0-continuous semigroup of contractions on $L^2(\pi)$.

Denote by $\mathcal{L}$ the generator of the semigroup Q_s on $L^2(\pi)$. Recall that $\mathcal{P}$ is the set of all polynomials. The following is proved in Section 4.4.

Proposition 4.2. *The set $\mathcal{P}$ is a common core of both $\mathcal{L}$ and L. In addition, we have*

$$\mathcal{L}F = LF + \sum_{j=1}^d v_j D_j F, \quad F \in \mathcal{P}. \tag{4.3}$$

We know from (4.3), (3.17) and the fact that v is divergence free (cf. [16, Corollary 12.22])

$$-\langle \mathcal{L}F, F \rangle_{L^2(\pi)} = -\langle LF, F \rangle_{L^2(\pi)} \geq \alpha_* \|F\|^2_{L^2(\pi)}, \quad F \in \mathfrak{H}_\infty, \quad F \perp 1. \tag{4.4}$$

This implies the exponential stability of the semigroup in $L^2(\pi)$:

$$\Big\| Q_t F - \int_{\mathcal{E}} F d\pi \Big\|_{L^2(\pi)} \leq e^{-\alpha_* t} \Big\| F - \int_{\mathcal{E}} F d\pi \Big\|_{L^2(\pi)}, \quad F \in L^2(\pi), \, t \geq 0. \tag{4.5}$$

Combining (4.2) with (4.5) we conclude, via an interpolation between $L^2(\pi)$ and $L^1(\pi)$, that

$$\Big\| Q_t F - \int_{\mathcal{E}} F d\pi \Big\|_{L^p(\pi)} \leq e^{-2\alpha_*(1-1/p)t} \Big\| F - \int_{\mathcal{E}} F d\pi \Big\|_{L^2(\pi)}, \quad F \in L^p(\pi), \, t \geq 0, \, p \in [1, 2], \tag{4.6}$$

and by interpolation between $L^2(\pi)$ and $L^\infty(\pi)$ also that

$$\Big\| Q_t F - \int_{\mathcal{E}} F d\pi \Big\|_{L^p(\pi)} \leq e^{-2\alpha_* t/p} \Big\| F - \int_{\mathcal{E}} F d\pi \Big\|_{L^2(\pi)}, \quad F \in L^p(\pi), \, t \geq 0, \, p \in [2, +\infty). \tag{4.7}$$

Theorem 4.3. *Suppose that $F_* \in L^p(\pi)$ for some $p \in (1, +\infty)$ and $\int_{\mathcal{E}} F d\pi = 0$. Then, the equation*

$$-\mathcal{L}\theta = F_* \tag{4.8}$$

admits a unique zero mean solution θ that belongs to the domain of the L^p-generator $\mathcal{L}$. In addition, if $F_* \in \mathfrak{H}_s$ for some $s > 0$, then $\theta \in \mathfrak{H}_{s+2}$.

The proof of this theorem is in Section 4.4. Since each $v_j \in \mathfrak{H}_\infty$, as an immediate consequence of Theorem 4.3, we conclude the following.

Corollary 4.4. *The equation*

$$-\mathcal{L}\chi_j = v_j, \quad j = 1, \dots, d, \tag{4.9}$$

admits a unique solution $\chi_j \in \mathcal{D}(\mathcal{L}) \cap \mathfrak{H}_\infty$ and $\chi_j \perp 1$ for each $j = 1, \dots, d$.

The solutions of (4.9) are known as the correctors. They can be used to express the effective diffusivity matrix appearing in the homogenized equation (2.7):

$$a_{ij} := \langle v_i, \chi_j \rangle_{L^2(\pi)} = \mathcal{E}_L(\chi_i, \chi_j), \quad i, j, = 1, \dots, d, \tag{4.10}$$

with

$$\mathcal{E}_L(F, G) := -\langle LF, G \rangle_{L^2(\pi)} = \int_{\mathcal{E}} \langle A\mathcal{D}F, \mathcal{D}G \rangle_H d\pi. \tag{4.11}$$

We define the corrector fields as stationary in (t, x) random fields $\tilde{\chi}_j : \mathbb{R}^{1+d} \times \Omega \to \mathbb{R}$, given by

$$\tilde{\chi}_j(t, x; w) := \chi_j(\tau_x w_t), \quad (t, x) \in \mathbb{R}^{1+d},\ j = 1, \dots, d. \tag{4.12}$$

Combining the results of Theorems 4.3 and 3.1, we conclude the following

Corollary 4.5. *The fields $\tilde{\chi}_j$, $\nabla_x \tilde{\chi}_j$ are square integrable for each $j = 1, \dots, d$:*

$$\sum_{i=1}^{d} \left\{ \mathbb{E}\tilde{\chi}_i^2(0,0) + \mathbb{E}\Big[|\nabla_x \tilde{\chi}_i(0,0)|^2\Big] \right\} < +\infty. \tag{4.13}$$

4.2 The "far away" independence

In order to deal with the spatial decorrelation properties of the velocity field, note that for each $x \in \mathbb{R}^d$ fixed, the set $\{\mathfrak{e}_n^x := \tau_{-x}\mathfrak{e}_n\}$ is an orthonormal base on H, and we can write

$$\mathfrak{e}_n^x = \sum_{m=1}^{+\infty} u_{nm}(x)\mathfrak{e}_m.$$

Here, $[u_{nm}(x)]$ is an infinite orthogonal matrix with

$$u_{nm}(x) = \langle \mathfrak{e}_n^x, \mathfrak{e}_m \rangle_H = \int_{\mathbb{R}^d} e^{-ik\cdot x}\hat{\mathfrak{e}}_n(k) \cdot \hat{\mathfrak{e}}_m^*(k)\sigma(k)dk, \quad n, m \geq 1. \tag{4.14}$$

As $\sigma(k)$ is compactly supported, each u_{nm} is bounded and analytic. We also have

$$u_{nm}(0) = \delta_{m,n}, \ u_{nm}(-x) = u_{mn}(x),$$

and

$$\sum_{k=1}^{+\infty} u_{nk}(x)u_{km}(y) = u_{nm}(x+y), \quad x, y \in \mathbb{R}^d,\ m, n \geq 1. \tag{4.15}$$

We will use the following "decorrelation lemma".

Proposition 4.6. *Suppose that α, σ satisfy the assumptions in Section 2, for any $n, m \geq 1$, let*

$$v_{nm}(x) := \langle A\mathfrak{e}_n^x, \mathfrak{e}_m\rangle_H, \tag{4.16}$$

then

$$\lim_{|x|\to+\infty} v_{nm}(x) = 0. \tag{4.17}$$

Proof. We have

$$v_{nm}(x) = \langle A\tau_{-x}\mathfrak{e}_n, \mathfrak{e}_m\rangle_H = \int_{\mathbb{R}^d} e^{-ik\cdot x}\hat{\mathfrak{e}}_n(k)\hat{\mathfrak{e}}_m^*(k)\alpha(k)\sigma(k)dk.$$

The result is an immediate consequence of the Riemann-Lebesgue lemma. □

4.3 The Itô formula for the environment process

To obtain the Itô formula for η_t, suppose that $\varphi \in H$ and $(t,x) \in \mathbb{R}^{1+d}$ and let

$$\hat{B}_s^{t,x}(\varphi) := \int_t^s \langle \tau_{-X^{t,x}(\sigma)}\varphi, dB_\sigma\rangle_H.$$

Define

$$\hat{B}_j^{t,x}(s) := \hat{B}_s^{t,x}(\mathfrak{e}_j) = \sum_{k=1}^{+\infty}\int_t^s u_{jk}\left(X^{t,x}(\sigma)\right)dB_k(\sigma), \quad j = 1, 2, \ldots, \tag{4.18}$$

where, as we recall $B_j(t) := B_t(\mathfrak{e}_j)$. The following result holds.

Corollary 4.7. *The space of polynomials $\mathcal{P}$ is a common core of $\mathcal{D}(L)$ and $\mathcal{D}(\mathcal{L})$. In addition, for any $F \in \mathcal{D}(\mathcal{L})$ we have $F \in \mathcal{D}(\mathcal{E}_L)$ and*

$$\langle(-\mathcal{L})F, F\rangle_{L^2(\pi)} = \mathcal{E}_L(F). \tag{4.19}$$

In addition, for any $F \in \mathcal{D}(\mathcal{L})$ and $(t,x) \in \mathbb{R}^{1+d}$ the following Itô formula holds

$$F(\eta_s^{t,x}) = F(\tau_x V_t) + \int_t^s \mathcal{L}F(\eta_\sigma^{t,x})d\sigma + \sqrt{2}\hat{M}_s^{t,x}(F), \tag{4.20}$$

where $\left(M_s^{t,x}(F)\right)_{s\geq t}$ is a continuous square integrable martingale given by

$$\hat{M}_s^{t,x}(F) := \sum_{j=1}^{+\infty}\int_t^s \mathcal{D}_j F(\eta_\sigma^{t,x})d\hat{B}_j^{t,x}(\sigma) = \int_t^s \left\langle \tau_{-X^{t,x}(\sigma)}\mathcal{D}F(\eta_\sigma^{t,x}), dB_\sigma\right\rangle_H, \tag{4.21}$$

with $\mathcal{D}_j$, $j = 1, 2, \ldots$ and $\mathcal{D}$ given by (3.25) and (3.24), respectively.

Proof. The first part follows from Proposition 4.2. Formula (4.19) holds for $F \in \mathcal{P}$, as can be easily seen by an application of (4.3). The extension to $D(\mathcal{L})$ can be done by an approximation.

For any $F \in \mathcal{P}$ the formula (4.21) follows from (3.34) and the definition of the process $\left(\eta_s^{t,x}\right)_{s\geq t}$, see (4.1). The extension to an arbitrary $F \in D(\mathcal{L})$ can, again, be achieved by an approximation argument. □

4.4 Proofs of Proposition 4.2 and Theorem 4.3

Proof of Proposition 4.2

Since $\mathcal{P}$ is dense in $L^2(\pi)$ and invariant under the semigroup P_t, it is a core of $\mathcal{D}(L)$. By a direct calculation using the Itô formula (3.33), it can be checked that $\mathcal{P} \subset \mathcal{D}(\mathcal{L})$ and the action of $\mathcal{L}$ on $F \in \mathcal{P}$ is given by (4.3). In what follows, we verify that in fact $\mathfrak{H}_4 \subset \mathcal{D}(\mathcal{L})$ and (4.3) holds also for any $F \in \mathfrak{H}_4$. Then, (4.4) also holds for all $F \in \mathfrak{H}_4$, so in particular $\mathcal{L}$ is dissipative on $\mathcal{P}$, i.e. for any $\lambda > 0$ we have $\|(\lambda - \mathcal{L})F\|_{L^2(\pi)} \ge \lambda \|F\|_{L^2(\pi)}$, $F \in \mathcal{P}$. Using Theorem 2.12, p. 16 of [10] we conclude that $\tilde{\mathcal{L}}$, the closure of $\mathcal{L}$, restricted to $\mathcal{P}$, is a generator of a strongly continuous semigroup on $L^2(\pi)$. But $\mathcal{L}$ itself is closed (as a generator of a C_0-semigroup) therefore $\tilde{\mathcal{L}} \subset \mathcal{L}$. The latter in turn implies that $\tilde{\mathcal{L}} = \mathcal{L}$, as then we have $(\lambda - \tilde{\mathcal{L}})^{-1} = (\lambda - \mathcal{L})^{-1}$ for any $\lambda > 0$. In particular, the above means that $\mathcal{P}$ is a core of $\mathcal{L}$, which ends the proof of Proposition 4.2. It remains to show that (4.3) holds for $F \in \mathfrak{H}_4$ and the density of $(\lambda - \mathcal{L})(\mathcal{P})$ in $L^2(\pi)$.

Recall that $F \in \mathfrak{H}_4$ iff

$$\sum_{n=0}^{+\infty} (n+1)^4 \|\mathfrak{p}_n F\|^2_{L^2(\pi)} < +\infty. \tag{4.22}$$

Thanks to (3.17), we conclude that

$$\|L\mathfrak{p}_n F\|_{L^2(\pi)} \le A_* n \|\mathfrak{p}_n F\|_{L^2(\pi)}, \quad n = 1, 2, \ldots \tag{4.23}$$

Let

$$F_n := \sum_{k=0}^{n} \mathfrak{p}_k F, \tag{4.24}$$

then $F_n \in \mathcal{D}(\mathcal{L})$ and

$$\mathcal{L}F_n = LF_n + \sum_{j=1}^{2} v_j D_j F_n.$$

Using the fact that $F \in \mathfrak{H}_4$ and (4.23), we conclude that $LF_n \to LF$, as $n \to +\infty$. Next, we show that $v_j D_j F_n$ converges in $L^2(\pi)$ for each $j = 1, \ldots, d$. Thanks to the first formula in (3.23) we have

$$\mathfrak{p}_m \left(v_j D_j \mathfrak{p}_k F_n \right) = 0, \quad |k - m| \ne 1.$$

Hence, for $n' > n$, using orthogonality we have

$$\begin{aligned} \Big\| \sum_{m,k=0}^{+\infty} \mathfrak{p}_m \left(v_j D_j \mathfrak{p}_k (F_{n'} - F_n) \right) \Big\|_{L^2(\pi)} \le & \Big\{ \sum_{m=0}^{+\infty} \|\mathfrak{p}_m \left(v_j D_j \mathfrak{p}_{m+1} (F_{n'} - F_n) \right)\|^2_{L^2(\pi)} \Big\}^{1/2} \\ & + \Big\{ \sum_{m=1}^{+\infty} \|\mathfrak{p}_{m+1} \left(v_j D_j \mathfrak{p}_m (F_{n'} - F_n) \right)\|^2_{L^2(\pi)} \Big\}^{1/2}. \end{aligned} \tag{4.25}$$

Using the Hölder inequality, we conclude that for $m > 1$

$$\|\mathfrak{p}_m \left(v_j D_j \mathfrak{p}_{m+1} (F_{n'} - F_n) \right)\|_{L^2(\pi)} \le \|v_j\|_{L^{2m}(\pi)} \|D_j \mathfrak{p}_{m+1} (F_{n'} - F_n)\|_{L^{2m/(m-1)}(\pi)}.$$

Since v_p is Gaussian, we have $\|v_p\|_{L^{2m}(\pi)} \sim (m!)^{1/(2m)}$, which, by virtue of Stirling's formula, is of the order $\sqrt{m}$. On the other hand, by the hypercontractivity of L^p norms with respect to a Gaussian measure, see e.g. Theorem 5.10 of [14], we have

$$\|D_j \mathfrak{p}_{m+1} (F_{n'} - F_n)\|_{L^{2m/(m-1)}(\pi)} \le \left(\frac{m+1}{m-1} \right)^{m/2} \|D_j \mathfrak{p}_{m+1} (F_{n'} - F_n)\|_{L^2(\pi)}.$$

Therefore, there exists $C > 0$ such that

$$
\begin{aligned}
&\|\mathfrak{p}_m\left(v_j D_j \mathfrak{p}_{m+1}(F_{n'} - F_n)\right)\|_{L^2(\pi)} \le C\sqrt{m}\,\|D_j \mathfrak{p}_{m+1}(F_{n'} - F_n)\|_{L^2(\pi)} \\
&\le C\sqrt{m}(m+1)\,\|\mathfrak{p}_{m+1}(F_{n'} - F_n)\|_{L^2(\pi)}, \quad m \ge 1,
\end{aligned}
$$

by virtue of parts (i) and (ii) of Theorem 3.1. A similar estimate holds for the second term in the right hand side of (4.25). As a result, there exists $C > 0$ such that

$$
\Big\|\sum_{j=1}^{d} v_j D_j (F_{n'} - F_n)\Big\|_{L^2(\pi)} \le C\|F_{n'} - F_n\|_{\mathfrak{H}_3}, \quad n' > n.
$$

The above implies that $\mathcal{L}F_n$ converges in $L^2(\pi)$. Thus, the right side of (4.3) makes sense for any $F \in \mathfrak{H}_4$, so that $F \in \mathcal{D}(\mathcal{L})$ and the action of $\mathcal{L}$ on $\mathfrak{H}_4$ is given by (4.3).

To show the density of $(\lambda - \mathcal{L})(\mathcal{P})$ in $L^2(\pi)$ we observe first that $(\lambda - \mathcal{L})(\mathfrak{H}_\infty)$ is dense in $L^2(\pi)$. Indeed, Lemma 2.21, p. 63 of [16] implies that given any $G \in \mathcal{P}$, there exists $F \in \mathfrak{H}_\infty$ such that it satisfies the resolvent equation $(\lambda - \mathcal{L})F = G$. Given $F \in \mathfrak{H}_\infty$ we let $F_n \in \mathcal{P}$ be defined by (4.24). The previous argument shows that $F_n \to F$ and $\mathcal{L}F_n \to \mathcal{L}F$, as $n \to +\infty$, in $L^2(\pi)$ (it even holds for $F \in \mathfrak{H}_4$). This proves that the closure of $(\lambda - \mathcal{L})(\mathcal{P})$ equals $L^2(\pi)$. The proof of Proposition 4.2 is therefore complete. □

Proof of Theorem 4.3

The zero mean solution of (4.8) is given by

$$
\theta = \int_0^{+\infty} Q_t F_* dt,
$$

and the integral in the right side converges, thanks to (4.5).

In light of the already proved Proposition 4.2, the generator $\mathcal{L}$ can be written on its core $\mathcal{P}$ as $L + \mathcal{A}$, where L is the generator of V_t, that is essentially self-adjoint on $\mathcal{P}$ (which is its core), and $\mathcal{A}F = \sum_{p=1}^{d} v_p D_p F$, $F \in \mathcal{P}$, is antisymmetric. We see from (3.18) and (3.19) that there exists a constant $C > 0$ such that

$$
|\langle \mathcal{A}F, G\rangle_{L^2(\pi)}| \le \sum_{p=1}^{d} \|D_p F\|_{L^2(\pi)} \|v_p G\|_{L^2(\pi)} \le C(n+1)^{1/2} \mathcal{E}_L^{1/2}(F)\mathcal{E}_L^{1/2}(G)
$$

for any $F \in \mathcal{H}_n$ and $G \in \mathcal{H}_{n+1}$, or $G \in \mathcal{H}_n$ and $F \in \mathcal{H}_{n+1}$, and $n = 0, 1, \ldots$.

If $F_* \in \mathfrak{H}_s$ for some $s > 0$ then

$$
\sum_{n=0}^{+\infty} (n+1)^{s+1} \|\mathfrak{p}_n F_*\|_{-1}^2 < +\infty,
$$

where

$$
\|\mathfrak{p}_n F_*\|_{-1}^2 := \sup_G \Big[2\langle \mathfrak{p}_n F_*, G\rangle_{L^2(\pi)} - \mathcal{E}_L(G)\Big] \asymp \frac{\|\mathfrak{p}_n F_*\|_{L^2(\pi)}^2}{n+1}, \quad n = 1, 2, \ldots,
$$

by virtue of part (i) of Theorem 3.1. By virtue of Lemma 2.21, p. 67 of [16], for any $s \ge 1$ the zero-mean solution of (4.8) satisfies

$$
\sum_{n=0}^{+\infty} (n+1)^{s+1} \mathcal{E}_L(\mathfrak{p}_n \theta) < +\infty,
$$

which, by another application of Theorem 3.1, shows that $\theta \in \mathfrak{H}_{s+2}$ (as $\mathcal{E}_L(\mathfrak{p}_n F_*) \asymp n\|\mathfrak{p}_n F_*\|_{L^2(\pi)}^2$), which ends the proof of the theorem. □

5 Proof of Theorem 2.1

To avoid cumbersome notations we consider only the case $N = 2$, as the general case can be argued using the same proof as below. Theorem 2.1 is an immediate corollary of the following result.

Theorem 5.1. *For any $(t, x, y) \in \mathbb{R} \times \mathbb{R}^{2d}$ with $x \neq y$, the processes $\big(X_\varepsilon^{t,x}(s), X_\varepsilon^{t,y}(s)\big)_{s \geq t}$ converge weakly over $C([t, +\infty); \mathbb{R}^{2d})$ to $\big(x + \beta_{s-t}, y + \beta'_{s-t}\big)_{s \geq t}$, where $(\beta_t)_{t \geq 0}$, $(\beta'_t)_{t \geq 0}$ are two independent copies of Brownian motion with the covariance matrix as in (2.6).*

This result is not very surprising – two particles starting at two different positions will see "nearly independent" environments. However, as the realizations of the velocity field in our case are analytic in space, the argument is slightly more delicate than, say, for velocity fields with finite range dependence, and relies on Proposition 4.6 rather than the usual mixing properties. As we have mentioned, convergence of each individual trajectory to a Brownian path is well known under our assumptions.

Decomposition of the trajectory

Let

$$\eta_{\varepsilon,s}^{t,x} := \tau_{X_\varepsilon^{t,x}(s)/\varepsilon} V_{s/\varepsilon^2},$$

then, using (4.9) and (4.20), we can decompose the $k-$th component of $X_\varepsilon^{t,x}(s)$, denoted by $X_{k,\varepsilon}^{t,x}(s)$, as

$$\begin{aligned} X_{k,\varepsilon}^{t,x}(s) &= x_k + \frac{1}{\varepsilon}\int_t^s v_k(\eta_{\varepsilon,\sigma}^{t,x})d\sigma = x_k - \frac{1}{\varepsilon}\int_t^s \mathcal{L}\chi_k(\eta_{\varepsilon,\sigma}^{t,x})d\sigma \\ &= x_k + \varepsilon Y_{k,\varepsilon}(t,s) + \sqrt{2}\int_t^s \langle \tau_{-X_\varepsilon^{t,x}(\sigma)/\varepsilon} \mathcal{D}\chi_k(\eta_{\varepsilon,\sigma}^{t,x}), dB_\sigma^\varepsilon\rangle_H = x_k + \varepsilon Y_{k,\varepsilon}(t,s) + \sqrt{2} M_{\varepsilon,k}^x(t,s), \end{aligned} \tag{5.1}$$

where

$$Y_{k,\varepsilon}(t,s) := \chi_k(\tau_{x/\varepsilon}V_{t/\varepsilon^2}) - \chi_k(\eta_{\varepsilon,s}^{t,x}), \quad M_{\varepsilon,k}^x(t,s) := \sum_{j=1}^{+\infty}\int_t^s \mathcal{D}_j\chi_k(\eta_{\varepsilon,\sigma}^{t,x})d\hat{B}_{j,\varepsilon}^{t,x}(\sigma),$$

and $\hat{B}_{j,\varepsilon}$ are defined using the change of variables (4.14) and (4.18)

$$\hat{B}_{j,\varepsilon}^{t,x}(s) := \sum_{k=1}^{+\infty}\int_t^s u_{jk}\Big(\frac{X_\varepsilon^{t,x}(\sigma)}{\varepsilon}\Big)dB_k^\varepsilon(\sigma), \;\; B_j^\varepsilon(t) := \varepsilon B_j(t/\varepsilon^2), \quad j = 1, 2, \dots \tag{5.2}$$

By Corollary 4.5, the main contribution to $X_{k,\varepsilon}^{t,x}(s)$ in (5.1) comes from $M_{\varepsilon,k}^x(t,s)$. In fact, one can show the following.

Proposition 5.2. *For any $(t, x) \in [0, T] \times \mathbb{R}^d$ and $\delta > 0$ we have*

$$\lim_{\varepsilon\to 0+} \mathbb{P}\left[\varepsilon \sup_{t\leq s\leq T} |\chi_k(\eta_{\varepsilon,s}^{t,x})| \geq \delta\right] = 0, \quad k = 1, \dots, d. \tag{5.3}$$

Proof. Due to stationarity, it suffices only to show that for each $k = 1, \dots, d$

$$\lim_{\varepsilon\to 0+} \varepsilon \sup_{0\leq s\leq T/\varepsilon^2} |\chi_k(\eta_s)| = \lim_{\varepsilon\to 0+}\left\{\varepsilon^2 \max_{0\leq k\leq [T/\varepsilon^2]} \mathcal{X}_k\right\}^{1/2} = 0, \quad \mathbb{P} \quad \text{a.s.}, \tag{5.4}$$

where $\mathcal{X}_k := \sup_{k\le s\le k+1} |\chi_k(\eta_s)|^2$. The sequence is stationary and ergodic, thanks to the results of Section 4.1. The Ito formula (4.20) applied to χ_k implies that

$$\mathbb{E}\mathcal{X}_1 < +\infty. \tag{5.5}$$

We claim that

$$\lim_{N\to+\infty} \frac{1}{N} \max_{1\le k\le N} \mathcal{X}_k = 0, \quad \mathbb{P} \quad \text{a.s.}, \tag{5.6}$$

which in turn yields (5.4).

Indeed, note first that $\mathcal{X}_N/N \to 0$, $\mathbb{P}$ a.s. Indeed, by the stationarity and ergodicity of the sequence $(\mathcal{X}_N)_{N\ge1}$ and the Birkhoff individual ergodic theorem we have

$$\frac{\mathcal{X}_N}{N} = \frac{1}{N}\sum_{k=1}^{N} \mathcal{X}_k - \frac{N-1}{N}\left(\frac{1}{N-1}\sum_{k=1}^{N-1} \mathcal{X}_k\right) \to 0, \quad \mathbb{P} \quad \text{a.s.}$$

Then

$$\lim_{N\to+\infty} \frac{1}{N} \max_{1\le k\le N} \mathcal{X}_k = \lim_{N\to+\infty} \max_{1\le k\le N} \left[\frac{k}{N}\cdot\frac{\mathcal{X}_k}{k}\right] = 0, \quad \mathbb{P} \quad \text{a.s.} \tag{5.7}$$

□

Decorrelation properties for separated trajectories

Next, we show that if trajectories are "slightly separated" then they have a small co-variation in a certain sense. We assume without loss of generality that

$$y = 0, \quad t = 0,$$

and set

$$M_\varepsilon^x(s) = (M_{\varepsilon,1}^x(s), \dots, M_{\varepsilon,d}^x(s)), \quad M_{\varepsilon,k}^x(s) := M_{\varepsilon,k}^x(0,s).$$

Let $\mathcal{Q}_x^\varepsilon$ be the joint law of $(X_\varepsilon^{0,x}(s), X_\varepsilon(s))_{s\ge0}$ over $\mathcal{C}_{2d} := C([0,+\infty);\mathbb{R}^{2d})$, where $X_\varepsilon(s) = X_\varepsilon^{0,0}(s)$. We know that each of the components $X_\varepsilon^{0,x}(s)$ and $X_\varepsilon(s)$ converges to a Brownian motion, so that the marginals of $\mathcal{Q}_x^\varepsilon$ form a tight family of measures on $\mathcal{C}_d$, thus $\mathcal{Q}_x^\varepsilon$ is also a tight family. In light of (5.1) and Proposition 5.2, the family $\tilde{\mathcal{Q}}_x^\varepsilon$ of the laws of $(M_\varepsilon^x(s), M_\varepsilon(s))_{s\ge0}$ are also tight, as $\varepsilon \downarrow 0$, and the families $\mathcal{Q}_x^\varepsilon$ and $\tilde{\mathcal{Q}}_x^\varepsilon$ have the same limiting points as $\varepsilon \downarrow 0$, so that we can focus on $\tilde{\mathcal{Q}}_x^\varepsilon$.

The processes $\hat{B}_{j,\varepsilon}^{t,x}(s)$ are square integrable, continuous trajectory martingales. Thanks to the expressions

$$c_{i,j} = \langle A\mathfrak{e}_i, \mathfrak{e}_j\rangle_H, \quad v_{n,m}(x) = \langle A\mathfrak{e}_n^x, \mathfrak{e}_m\rangle_H,$$

as well as stationarity in space, their co-variations are

$$\begin{aligned}\langle \hat{B}_{j_1,\varepsilon}^{t,x}, \hat{B}_{j_2,\varepsilon}^{t,y}\rangle_s &= \sum_{k,m=1}^{+\infty} c_{k,m} \int_t^s u_{j_1,k}\Big(\frac{X_\varepsilon^{t,x}(\sigma)}{\varepsilon}\Big) u_{j_2,m}\Big(\frac{X_\varepsilon^{t,y}(\sigma)}{\varepsilon}\Big) d\sigma \\ &= \int_t^s v_{j_1,j_2}\Big(\frac{X_\varepsilon^{t,x}(\sigma) - X_\varepsilon^{t,y}(\sigma)}{\varepsilon}\Big) d\sigma, \quad s\ge t,\ x,y\in\mathbb{R}^d, \end{aligned} \tag{5.8}$$

so we have

$$\langle M_{\varepsilon,k}^x, M_{\varepsilon,\ell}^0\rangle_s = \sum_{j,m=1}^{+\infty} \int_0^s v_{j,m}\Big(\frac{X_\varepsilon^{0,x}(\sigma) - X_\varepsilon(\sigma)}{\varepsilon}\Big) \mathcal{D}_j\chi_k(\eta_{\varepsilon,\sigma}^{0,x})\mathcal{D}_m\chi_\ell(\eta_{\varepsilon,\sigma}) d\sigma = \int_0^s m_{k,\ell}^{\varepsilon,x,0}(\sigma) d\sigma,$$

with

$$m^{\varepsilon,x,y}_{k,\ell}(\sigma) := \left\langle A\mathcal{D}\chi_k(\eta^{0,x}_{\varepsilon,\sigma}), \tau_{[X^{0,x}_\varepsilon(\sigma)-X^{0,y}_\varepsilon(\sigma)]/\varepsilon}\mathcal{D}\chi_\ell\left(\eta^{0,y}_{\varepsilon,\sigma}\right)\right\rangle_H,\ k,\ell = 1,\ldots,d. \tag{5.9}$$

We now perform a finite-dimensional approximation: given $N \in \mathbb{N}$, let $\mathcal{D}\chi^N_k = \sum_{j=1}^N \mathcal{D}_j\chi_k\mathfrak{e}_j$. As

$$\sum_{k=1}^d \int_{\mathcal{E}} \|\mathcal{D}\chi_k\|^2_H d\pi < +\infty,$$

we have

$$\lim_{N\to+\infty} \sum_{k=1}^d \int_{\mathcal{E}} \|\mathcal{D}\chi_k - \mathcal{D}\chi^N_k\|^2_H d\pi = \lim_{N\to+\infty} \sum_{k=1}^d \sum_{j=N+1}^{+\infty} \int_{\mathcal{E}} |\mathcal{D}_j\chi_k|^2 d\pi = 0. \tag{5.10}$$

Define $m^{\varepsilon,x,y}_{k,\ell,N}$ by (5.9), with $\mathcal{D}\chi_k$, $\mathcal{D}\chi_\ell$ replaced by $\mathcal{D}\chi^N_k$, $\mathcal{D}\chi^N_\ell$ correspondingly. Recall that

$$\eta^{t,x}_s = \tau_{X^{t,x}(s)}V_s, \quad \eta^{t,x}_{\varepsilon,s} = \tau_{X^{t,x}_\varepsilon(s)/\varepsilon}V_{s/\varepsilon^2},$$

so we have

$$\eta^{t,x}_{\varepsilon,s} = \eta^{t/\varepsilon^2,x/\varepsilon}_{s/\varepsilon^2}.$$

The following approximation property holds.

Lemma 5.3. *For any $(s,x) \in [0,+\infty) \times \mathbb{R}^d$, we have*

$$\lim_{N\to+\infty} \sup_{\varepsilon\in(0,1]} \mathbb{E}\left|m^{\varepsilon,x,0}_{k,\ell}(s) - m^{\varepsilon,x,0}_{k,\ell,N}(s)\right| = 0,\ k,\ell = 1,\ldots,d. \tag{5.11}$$

Proof. The expression under the limit in (5.11) can be estimated using the Cauchy-Schwarz inequality as

$$\begin{aligned} &C\Big\{\mathbb{E}\Big\|\left(\mathcal{D}\chi_k - \mathcal{D}\chi^N_k\right)(\eta^{0,x/\varepsilon}_{s/\varepsilon^2})\Big\|^2_H\Big\}^{1/2}\Big\{\mathbb{E}\Big\|\tau_{X^{0,x/\varepsilon}(s/\varepsilon^2)-X(s/\varepsilon^2)}\mathcal{D}\chi_\ell(\eta_{s/\varepsilon^2})\Big\|^2_H\Big\}^{1/2} \\ &+ C\Big\{\mathbb{E}\Big\|\tau_{X^{0,x/\varepsilon}(s/\varepsilon^2)-X(s/\varepsilon^2)}\mathcal{D}\chi^N_k(\eta_{s/\varepsilon^2})\Big\|^2_H\Big\}^{1/2}\Big\{\mathbb{E}\Big\|\left(\mathcal{D}\chi_\ell - \mathcal{D}\chi^N_\ell\right)(\eta^{0,x/\varepsilon}_{s/\varepsilon^2})\Big\|^2_H\Big\}^{1/2}, \end{aligned}$$

with a constant $C > 0$, independent of $\varepsilon > 0$ and N. The group τ_x is unitary on H and the processes $\eta^{0,x/\varepsilon}_{t/\varepsilon^2}$ are stationary in t for each x fixed. Therefore, the above expression equals

$$C\Big\{\int_{\mathcal{E}} \|\mathcal{D}\chi_k - \mathcal{D}\chi^N_k\|^2_H d\pi\Big\}^{1/2}\Big\{\int_{\mathcal{E}} \|\mathcal{D}\chi_\ell\|^2_H\, d\pi\Big\}^{1/2} + C\Big\{\int_{\mathcal{E}} \Big\|\mathcal{D}\chi^N_k\Big\|^2_H d\pi\Big\}^{1/2}\Big\{\int_{\mathcal{E}} \|\mathcal{D}\chi_\ell - \mathcal{D}\chi^N_\ell\|^2_H d\pi\Big\}^{1/2}.$$

The claim of the lemma can be now concluded directly from (5.10) and Corollary 4.5. □

The next lemma shows that if the trajectories are sufficiently far apart, their co-variation is small. For any measurable set $A \subset \Omega$ and random variable X, we write $\mathbb{E}[X, A] = \mathbb{E}[X1_A]$.

Lemma 5.4. *For any $\gamma \in (0,1)$, $x \neq 0$ and $s, N > 0$ we have*

$$\lim_{\varepsilon\to 0} \mathbb{E}\left[\left|m^{\varepsilon,x,0}_{k,\ell,N}(s)\right|,\ |X^{0,x}_\varepsilon(s) - X_\varepsilon(s)| \ge \varepsilon^\gamma\right] = 0,\ k,\ell = 1,\ldots,d. \tag{5.12}$$

Proof. We write (5.9) as

$$m^{\varepsilon,x,0}_{k,\ell,N}(s)=\sum_{j,p=1}^{N}\mathcal{D}_j\chi_k(\eta^{0,x/\varepsilon}_{s/\varepsilon^2})\mathcal{D}_p\chi_\ell(\eta_{s/\varepsilon^2})v_{j,p}\Big(\frac{X^{0,x}_\varepsilon(s)-X_\varepsilon(s)}{\varepsilon}\Big),\tag{5.13}$$

and estimate

$$\mathbb{E}\big[|m^{\varepsilon,x,N}_{k,\ell}(s)|,\,|X^{0,x}_\varepsilon(s)-X_\varepsilon(s)|\ge\varepsilon^\gamma\big]\le\sum_{j,p=1}^{N}\sup_{|x|\ge\varepsilon^{\gamma-1}}|v_{j,p}(x)|\mathbb{E}\big|\mathcal{D}_j\chi_k(\eta^{0,x/\varepsilon}_{s/\varepsilon^2})\mathcal{D}_p\chi_\ell(\eta_{s/\varepsilon^2})\big|$$

$$\le\sum_{j,p=1}^{N}\sup_{|x|\ge\varepsilon^{\gamma-1}}|v_{j,p}(x)|\|\mathcal{D}_j\chi_k\|_{L^2(\pi)}\|\mathcal{D}_p\chi_\ell\|_{L^2(\pi)}.$$

Now, the conclusion of the lemma follows from Proposition 4.6 since N is finite. □

The concatenated process

Let $\mathcal{M}_s$ be the natural filtration corresponding to the canonical process $(X(s),Y(s))_{s\ge0}$ on $\mathcal{C}_{2d}$, and $\mathcal{M}$ be the smallest σ-algebra generated by all $\mathcal{M}_s$, $s\ge0$. Fix $\gamma\in(0,1)$, and for any $(X,Y)\in\mathcal{C}_{2d}$ and $\varepsilon\ge0$ let

$$T_\varepsilon(X,Y):=\inf\{s\ge0:\,|X(s)-Y(s)|\le\varepsilon^\gamma\}.$$

We adopt the convention that the infimum of an empty set equals $+\infty$. Let us modify the processes $M^x_\varepsilon(s)$ and $M_\varepsilon(s)$ as follows:

$$\tilde{M}^x_\varepsilon(s):=\begin{cases}M^x_\varepsilon(s), & 0\le s\le\tilde{T}_\varepsilon,\\ M^x_\varepsilon(\tilde{T}_\varepsilon)+\beta_{s-\tilde{T}_\varepsilon}, & \tilde{T}_\varepsilon\le s,\end{cases}$$

$$\tilde{M}_\varepsilon(s):=\begin{cases}M_\varepsilon(s), & 0\le s\le\tilde{T}_\varepsilon,\\ M_\varepsilon(\tilde{T}_\varepsilon)+\tilde{\beta}_{s-\tilde{T}_\varepsilon}, & \tilde{T}_\varepsilon\le s.\end{cases}$$

Here $\tilde{T}_\varepsilon:=T_\varepsilon(X^x_\varepsilon,X_\varepsilon)$, and β_s and $\tilde{\beta}_s$ are two copies of the Brownian motion with the covariance given by (2.6) that are independent of each other and of $(X^x_\varepsilon(s),X_\varepsilon(s))_{s\ge0}$. We denote by $\hat{\mathcal{Q}}^\varepsilon_x$ the law of $(\tilde{M}^x_\varepsilon(s),\tilde{M}_\varepsilon(s))_{s\ge0}$ on $(\mathcal{C}_{2d},\mathcal{M})$, and the law of $(x+\beta_s,y+\tilde{\beta}_s)$ by $\mathfrak{Q}_{x,y}$.

The following proposition shows that the law $\hat{\mathcal{Q}}^\varepsilon_x$ becomes close to $\mathfrak{Q}_{x,0}$, as $\varepsilon\to0$. To abbreviate the notation, we set

$$N_t(G):=G(X(t),Y(t))-G(X(0),Y(0))-\int_0^t(A_x+A_y)G(X(\varrho),Y(\varrho))\,d\varrho$$

for any $G\in C^2_0(\mathbb{R}^{2d})$ and $t\ge0$. Here A_x, A_y denote the differential operators of the form

$$AG(x):=\frac{1}{2}\sum_{k,\ell=1}^{d}a_{k\ell}\partial^2_{x_k,x_\ell}G(x),\tag{5.14}$$

acting on the x and y variables respectively.

Proposition 5.5. *For any $x\in\mathbb{R}^d$, the family of laws $(\hat{\mathcal{Q}}^\varepsilon_x)_{\varepsilon\in(0,1]}$ is tight. Suppose, in addition, that $x\ne0$, $\zeta\in C_b((\mathbb{R}^{2d})^n)$, and $0\le t_1<\dots<t_n\le t<v\le T$. Then, we have*

$$\lim_{\varepsilon\to0}E^\varepsilon_x\left\{[N_v(G)-N_t(G)]\,\tilde{\zeta}\right\}=0\tag{5.15}$$

for any $G \in C_0^2(\mathbb{R}^{2d})$. Here E_x^ε denotes the expectation with respect to $\hat{\mathcal{Q}}_x^\varepsilon$, and

$$\tilde{\zeta}(X,Y) := \zeta(X(t_1),Y(t_1),\dots,X(t_n),Y(t_n)), \quad (X,Y) \in \mathcal{C}_{2d}.$$

Proof. Tightness is a direct consequence of the tightness of $\tilde{\mathcal{Q}}_x^\varepsilon$, $\varepsilon \in (0,1]$, so we only need to show (5.15). Denote

$$\hat{m}_{k,\ell}^{\varepsilon,x}(s) := \begin{cases} m_{k,\ell}^{\varepsilon,x,x}(s), & s \le \tilde{T}_\varepsilon, \\ a_{k\ell}, & \tilde{T}_\varepsilon < s, \end{cases} \qquad \tilde{m}_{k,\ell}^{\varepsilon}(s) := \begin{cases} m_{k,\ell}^{\varepsilon,x,0}(s), & s \le \tilde{T}_\varepsilon, \\ 0, & \tilde{T}_\varepsilon < s, \end{cases}$$

where $m_{k,\ell}^{\varepsilon,x,y}$ were defined in (5.9). Using the Itô formula, we conclude that

$$\mathcal{N}_t^\varepsilon(G) := G(\tilde{M}_\varepsilon^x(t),\tilde{M}_\varepsilon(t)) - G(\tilde{M}_\varepsilon^x(0),\tilde{M}_\varepsilon(0)) - \int_0^t (A_\varepsilon^x(s)G + A_\varepsilon^{x,0}(s)G + A_\varepsilon^0(s)G)(\tilde{M}_\varepsilon^x(s),\tilde{M}_\varepsilon(s))ds$$

is a martingale, where

$$A_\varepsilon^x(s)G(x,y) := \frac{1}{2}\sum_{k,\ell=1}^d \hat{m}_{k,\ell}^{\varepsilon,x}(s)\partial^2_{x_k,x_\ell}G(x,y),$$

$$A_\varepsilon^0(s)G(x,y) := \frac{1}{2}\sum_{k,\ell=1}^d \hat{m}_{k,\ell}^{\varepsilon,0}(s)\partial^2_{y_k,y_\ell}G(x,y),$$

and

$$A_\varepsilon^{x,0}(s)G(x,y) := \sum_{k,\ell=1}^d \tilde{m}_{k,\ell}^{\varepsilon}(s)\partial^2_{x_k,y_\ell}G(x,y).$$

Let

$$\tilde{\zeta}_\varepsilon := \zeta(\tilde{M}_\varepsilon^x,\tilde{M}_\varepsilon) \quad \text{and} \quad \tilde{\zeta}'_\varepsilon = \zeta(X_\varepsilon^{0,x},X_\varepsilon).$$

Since

$$\begin{aligned}
&\lim_{\varepsilon\to 0} E_x^\varepsilon\left\{[N_v(G)-N_t(G)]\,\tilde{\zeta}\right\} \\
&= \lim_{\varepsilon\to 0}\left\{\mathbb{E}\left\{\left[N_v(G;\tilde{M}_\varepsilon^x,\tilde{M}_\varepsilon) - N_t(G;\tilde{M}_\varepsilon^x,\tilde{M}_\varepsilon)\right]\tilde{\zeta}_\varepsilon\right\} - \mathbb{E}\left\{[\mathcal{N}_v^\varepsilon(G)-\mathcal{N}_t^\varepsilon(G)]\tilde{\zeta}_\varepsilon\right\}\right\},
\end{aligned}$$

to prove (5.15), it suffices to show that

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\mathbb{E}\left\{\left[\int_t^v (A_\varepsilon^x(\sigma)-A_x)\,G(\tilde{M}_\varepsilon^x(\sigma),\tilde{M}_\varepsilon(\sigma))d\sigma\right]\tilde{\zeta}_\varepsilon\right\} = 0, \\
&\lim_{\varepsilon\to 0}\mathbb{E}\left\{\left[\int_t^v \left(A_\varepsilon^0(\sigma)-A_y\right)G(\tilde{M}_\varepsilon^x(\sigma),\tilde{M}_\varepsilon(\sigma))d\sigma\right]\tilde{\zeta}_\varepsilon\right\} = 0, \qquad (5.16)\\
&\lim_{\varepsilon\to 0}\mathbb{E}\left\{\left[\int_t^v A_\varepsilon^{x,0}(\sigma)G(\tilde{M}_\varepsilon^x(\sigma),\tilde{M}_\varepsilon(\sigma))d\sigma\right]\tilde{\zeta}_\varepsilon\right\} = 0.
\end{aligned}$$

Choose an arbitrary $\delta > 0$ and integers $N_1, N_2 > 0$, and divide the interval $[t,v]$ into subintervals $[s_{k-1},s_k]$, with $s_k := t + k(v-t)/N_1$, $k = 0,\dots,N_1$. As the laws of $(\tilde{M}_\varepsilon^x,\tilde{M}_\varepsilon)$ are tight, we can choose N_1, N_2 sufficiently large so that the limit of the first expression in (5.16) differs only by δ from

$$\lim_{\varepsilon\to 0}\sum_{j=1}^{N_1}\mathbb{E}\left\{\left[\int_{s_{j-1}\wedge\tilde{T}_\varepsilon}^{s_j\wedge\tilde{T}_\varepsilon} w_{k,\ell}^{(N_2)}(\eta_{\varepsilon,\sigma}^{s_{j-1},X_\varepsilon^{0,x}(s_{j-1})/\varepsilon})\partial^2_{x_k,x_\ell}G(\tilde{M}_\varepsilon^x(s_{j-1}),\tilde{M}_\varepsilon(s_{j-1}))d\sigma\right]\tilde{\zeta}_\varepsilon\right\}, \qquad (5.17)$$

where

$$w_{k,\ell}^{(N_2)} := \langle A\mathcal{D}\mathfrak{P}_{0,N_2}\chi_k, \mathcal{D}\mathfrak{P}_{0,N_2}\chi_\ell\rangle_H - \int_{\mathcal{E}} \langle A\mathcal{D}\mathfrak{P}_{0,N_2}\chi_k, \mathcal{D}\mathfrak{P}_{0,N_2}\chi_\ell\rangle_H d\pi$$

and $\mathfrak{P}_{0,N_2} := \sum_{n=0}^{N_2} \mathfrak{p}_n$. Clearly $w_{k,\ell}^{(N_2)} \in \mathfrak{H}_\infty$. Let $\theta_{k,\ell}^{(N_2)} \in \mathfrak{H}_\infty$ be the mean-zero solutions of

$$-\mathcal{L}\theta_{k,\ell}^{(N_2)} = w_{k,\ell}^{(N_2)}, \quad k,\ell = 1,\ldots,d, \tag{5.18}$$

that exist, thanks to Theorem 4.3. Using formula (4.20) we get

$$d\theta_{k,\ell}^{(N_2)}(\eta_{\varepsilon,\sigma}^{s,x}) = -\frac{1}{\varepsilon^2}\mathcal{L}\theta_{k,\ell}^{(N_2)}(\eta_{\varepsilon,\sigma}^{s,x})d\sigma + \frac{1}{\varepsilon}\Big\langle \tau_{-X_\varepsilon^{s,x}(\sigma)/\varepsilon}\mathcal{D}\theta_{k,\ell}^{(N_2)}(\eta_{\varepsilon,\sigma}^{s,x}), dB_\sigma^\varepsilon\Big\rangle_H,$$

with $B_\sigma^\varepsilon := \varepsilon B_{\sigma/\varepsilon^2}$. Substituting from the above into (5.17), we conclude that

$$\mathbb{E}\left\{\left[\int_{s_{j-1}\wedge\tilde{T}_\varepsilon}^{s_j\wedge\tilde{T}_\varepsilon} \mathcal{L}\theta_{k,\ell}^{(N_2)}(\eta_{\sigma/\varepsilon^2}^{s_{j-1}/\varepsilon^2, X^{0,x/\varepsilon}(s_{j-1}/\varepsilon^2)})\partial_{x_k,x_\ell}^2 G(\tilde{M}_\varepsilon^x(s_{j-1}), \tilde{M}_\varepsilon(s_{j-1}))d\sigma\right]\tilde{\zeta}_\varepsilon\right\} = O(\varepsilon^2).$$

It follows that

$$\lim_{\varepsilon\to 0}\sum_{j=1}^{N_1}\mathbb{E}\left\{\left[\int_{s_{j-1}\wedge\tilde{T}_\varepsilon}^{s_j\wedge\tilde{T}_\varepsilon} \mathcal{L}\theta_{k,\ell}^{(N_2)}(\eta_{\sigma/\varepsilon^2}^{s_{j-1}/\varepsilon^2, X^{0,x/\varepsilon}(s_{j-1}/\varepsilon^2)})\partial_{x_k,x_\ell}^2 G(\tilde{M}_\varepsilon^x(s_{j-1}), \tilde{M}_\varepsilon(s_{j-1}))d\sigma\right]\tilde{\zeta}_\varepsilon\right\} = 0,$$

for any N_1, N_2 fixed, and the first equality in (5.16) follows. The second equality can be obtained in the same way. The third equality is then a direct consequence of Lemma 5.3 and 5.4. $\square$

It follows from Proposition 5.5 that $\hat{\mathcal{Q}}_x^\varepsilon$ converge weakly, as $\varepsilon \to 0$ to $\mathfrak{Q}_{x,0}$. We also have

$$\tilde{\mathcal{Q}}_x^\varepsilon(A, T_\varepsilon > T) = \hat{\mathcal{Q}}_x^\varepsilon(A, T_\varepsilon > T), \quad A \in \mathcal{M}_T, T > 0,\ \varepsilon \in (0,1]. \tag{5.19}$$

Therefore, for any $0 < \varepsilon < \varepsilon' \le 1$ we have

$$\tilde{\mathcal{Q}}_x^\varepsilon(T_\varepsilon \le T) = \hat{\mathcal{Q}}_x^\varepsilon(T_\varepsilon \le T) \le \hat{\mathcal{Q}}_x^\varepsilon(T_{\varepsilon'} \le T). \tag{5.20}$$

Passing to the limit, as $\varepsilon \to 0$, and using elementary properties of weak convergence of probability measures, we see that

$$\limsup_{\varepsilon\to 0} \hat{\mathcal{Q}}_x^\varepsilon(T_{\varepsilon'} \le T) \le \mathfrak{Q}_{x,0}(T_{\varepsilon'} \le T), \quad \varepsilon' \in (0,1]. \tag{5.21}$$

The last point is that, as β_t and $\tilde{\beta}_t$ are two independent Brownian motions with non-degenerate covariances, and $d \ge 2$, we have

$$\mathfrak{Q}_{x,0}(T_0 < T) = 0, \quad \text{for any } T > 0,\ x \ne 0. \tag{5.22}$$

The weak convergence of $\hat{\mathcal{Q}}_x^\varepsilon$ to $\mathfrak{Q}_{x,0}$ and (5.19)-(5.22) imply the conclusion of Theorem 5.1. $\square$

Proof of Corollary 2.2

It suffices to show that

$$\lim_{\varepsilon\to 0}\mathbb{E}\langle u_\varepsilon(t), \varphi\rangle = \langle \bar{u}(t), \varphi\rangle \quad \text{and} \quad \lim_{\varepsilon\to 0}\mathbb{E}\langle u_\varepsilon(t), \varphi\rangle^2 = \langle \bar{u}(t), \varphi\rangle^2. \tag{5.23}$$

The first equality follows from the weak convergence of $u_\varepsilon(t,x) = u_0\left(X_\varepsilon^{t,x}(T)\right)$ to $u_0(x+\beta_{T-t})$. To prove the second equality, observe that for any $\varphi \in C_0^\infty(\mathbb{R}^d)$

$$\lim_{\varepsilon\to 0} \mathbb{E}\langle u_\varepsilon(t), \varphi\rangle^2 = \int_{\mathbb{R}^{2d}} \left\{ \lim_{\varepsilon\to 0} \mathbb{E}\left[u_0\left(X_\varepsilon^{t,x}(T)\right) u_0\left(X_\varepsilon^{t,y}(T)\right)\right]\right\} \varphi(x)\varphi(y)dxdy.$$

Using Theorem 5.1 we conclude that the right side equals

$$\int_{\mathbb{R}^{2d}} \mathbb{E}\left[u_0(x+\beta_{T-t})u_0(y+\tilde\beta_{T-t})\right]\varphi(x)\varphi(y)dxdy = \langle \bar u(t), \varphi\rangle^2.$$

□

6 Proofs of Theorem 2.3 and Corollary 2.4

Let $\tilde f : D_T \times \mathcal{E} \to \mathbb{R}$ be given by

$$\tilde f(t,x,u,w) := f(t,x,u,w) - \bar f(t,x,u),$$

where $\bar f$ is defined by (2.13), and $\Theta : D_T\times\mathcal{E}\to\mathbb{R}$ be the unique solution of

$$-\mathcal{L}\Theta(t,x,u,w) = \tilde f(t,x,u,w), \quad \int_{\mathcal{E}} \Theta(t,x,u,w)d\pi = 0, \quad (t,x,u)\in D_T. \tag{6.1}$$

We note that $(t,x,u)\in D_T$ is fixed in (6.1), and the $\mathcal{L}$ operator is acting only on the variable $w\in\mathcal{E}$. To simplify the notation, we will keep the dependence on w implicit. By the Itô formula (4.20),

$$\int_{s_1}^{s_2} \tilde f(s,y,u,\eta^{t,x}_{\varepsilon,\sigma})d\sigma = \varepsilon^2 \Delta^\varepsilon_{s_1,s_2}\Theta(s,y,u) + \varepsilon\Delta^\varepsilon_{s_1,s_2} M_\varepsilon(s,y,u), \tag{6.2}$$

where

$$\Delta^\varepsilon_{s_1,s_2}\Theta(s,y,u) := \Theta(s,y,u,\eta^{t,x}_{\varepsilon,s_1}) - \Theta(s,y,u,\eta^{t,x}_{\varepsilon,s_2}),$$

$$\Delta^\varepsilon_{s_1,s_2}M(s,y,u) := \int_{s_1}^{s_2} \langle \tau_{-X_\varepsilon^{t,x}(\sigma)}\mathcal{D}\Theta(s,y,u,\eta^{t,x}_{\varepsilon,\sigma}), dB^\varepsilon_\sigma\rangle_H, \quad t\le s, s_1, s_2\le T,\ s_1<s_2,$$

and $B^\varepsilon_t := \varepsilon B(t/\varepsilon^2)$. To simplify the notation, we will omit the dependence of Θ on the w variable. Given any $N>0$, we let

$$K_N := [(y,u)\in\mathbb{R}^{d+1} : \max\{|y|, |u|\}\le N]. \tag{6.3}$$

Lemma 6.1. *For any $N>0$, $p\in(1,+\infty)$ and $t\le s_1<s_2\le T$, we have*

$$\sup_{s\in[t,T]} \left\langle \sup_{(y,u)\in K_N} |\Theta(s,y,u)|^p \right\rangle_\pi < +\infty, \tag{6.4}$$

and

$$\limsup_{\varepsilon\to 0} \sup_{s\in[t,T]} \left\langle \sup_{(y,u)\in K_N} |\Delta^\varepsilon_{s_1,s_2}M(s,y,u)|^2\right\rangle_\pi < +\infty. \tag{6.5}$$

Proof. Given an integer multi-index $m=(m_0,\dots,m_d)$, we denote by $|m| = \sum_{j=0}^d m_j$, and by ∇^m the mixed partial $\partial_u^{m_0}\partial_{y_1}^{m_1}\dots\partial_{y_d}^{m_d}$. Then $\nabla^m\Theta(t,y,u)$ is the unique π-zero-mean solution of

$$-\mathcal{L}\nabla^m\Theta(t,y,u) = \nabla^m\tilde f(t,y,u). \tag{6.6}$$

In addition, thanks to (4.6) and (4.7), for any $p \in (1, +\infty)$ there exists $C > 0$ such that

$$\|\nabla^m \Theta(t, y, u)\|_{L^p(\pi)} \le C \|\nabla^m \tilde{f}(t, y, u)\|_{L^p(\pi)}, \quad (t, y, u) \in D_T. \tag{6.7}$$

The Sobolev embedding theorem implies that for $p > d + 1$ there exists a constant $C > 0$ such that

$$\sup_{(y,u) \in K_N} |\Theta(t, y, u)|^p \le C \int_{K_N} \Big[|\Theta(t, y, u)|^p + \sum_{|m|=1} |\nabla^m \Theta(t, y, u)|^p \Big] dy du. \tag{6.8}$$

Integrating both sides of (6.8) in the w variable and using (6.7), we obtain (6.4).

Next, we proceed with the proof of (6.5). It is clear from the definition of the Malliavin derivative, see (3.25) and (3.24), that

$$\mathcal{D} \nabla^m \Theta(t, y, u) = \nabla^m \mathcal{D} \Theta(t, y, u),$$

and

$$\nabla^m (\Delta^\varepsilon_{s_1, s_2} M)(s, y, u) = \int_{s_1}^{s_2} \langle \tau_{-X^{t,x}_\varepsilon(\sigma)} \nabla^m \mathcal{D} \Theta(s, y, u; \eta^{t,x}_{\varepsilon,\sigma}), dB^\varepsilon_\sigma \rangle_H, \quad t \le s, s_1, s_2 \le T.$$

Again, thanks to the Sobolev embedding theorem, there exists a constant $C > 0$ such that

$$\sup_{(y,u) \in K_N} \big[\Delta^\varepsilon_{s_1, s_2} M(s, y, u)\big]^2 \le C \int_{K_N} \Big\{ \big[\Delta^\varepsilon_{s_1, s_2} M(s, y, u)\big]^2 + \sum_{|m|=n} \big[\nabla^m (\Delta^\varepsilon_{s_1, s_2} M)(s, y, u)\big]^2 \Big\} dy du, \tag{6.9}$$

provided that $n > (d + 1)/2$. Using the Itô isometry and the above estimate, we conclude that for any $s_1 < s_2$

$$\begin{aligned} \mathbb{E}\Big\{ \sup_{(y,u) \in K_N} \big[\Delta^\varepsilon_{s_1, s_2} M(s, y, u)\big]^2 \Big\} &\le C(s_2 - s_1) \int_{K_N} \Big\langle \|\mathcal{D}\Theta(s, y, u)\|^2_H + \sum_{|m|=n} \|\mathcal{D}\nabla^m \Theta(s, y, u)\|^2_H \Big\rangle_\pi dy du \\ &\le C(s_2 - s_1) \int_{K_N} \Big[\mathcal{E}_{\mathcal{L}} \left(\Theta(s, y, u) \right) + \sum_{|m|=n} \mathcal{E}_{\mathcal{L}} \left(\nabla^m \Theta(s, y, u) \right) \Big] dy du. \end{aligned} \tag{6.10}$$

This proves (6.5) in light of (6.7) and (2.12). □

Next, let

$$U_\varepsilon(s) := u_\varepsilon(s, X^{t,x}_\varepsilon(s)), \quad F_\varepsilon(s) := f(s, X^{t,x}_\varepsilon(s), U_\varepsilon(s), \eta^{t,x}_{\varepsilon,s}) \tag{6.11}$$

and $\tilde{F}_\varepsilon(s) := \tilde{f}(s, X^{t,x}_\varepsilon(s), U_\varepsilon(s), \eta^{t,x}_{\varepsilon,s})$, $t \le s \le T$. We have

$$u_\varepsilon(t, x) = U_\varepsilon(t)$$

and

$$U_\varepsilon(T) = U_\varepsilon(s) + \int_s^T F_\varepsilon(\sigma) d\sigma, \tag{6.12}$$

so that

$$U_\varepsilon(T) - U_\varepsilon(s) - \int_s^T \bar{f}(\sigma, X^{t,x}_\varepsilon(\sigma), U_\varepsilon(\sigma)) d\sigma = \int_s^T \tilde{F}_\varepsilon(\sigma) d\sigma. \tag{6.13}$$

The following lemma shows that the random fluctuation in the r.h.s. of the above display is negligible in the limit.

Proposition 6.2. *For any $\delta > 0$, $t \le s \le T$ we have*

$$\lim_{\varepsilon \to 0} \mathbb{P}\Big[\Big| \int_s^T \tilde{F}_\varepsilon(\sigma) d\sigma \Big| \ge \delta \Big] = 0. \tag{6.14}$$

Proof. Since f is bounded, the laws of the processes $(U_\varepsilon(s))_{t\le s\le T}$ are tight over $C[t,T]$, as $\varepsilon\to 0$. In consequence, the laws of the joint process $\big(X_\varepsilon^{t,x}(s),U_\varepsilon(s)\big)_{t\le s\le T}$ are also tight. Given any $\rho>0$, one can choose $N>0$ such that

$$\mathbb{P}\Big[\sup_{s\in[t,T]}\max[|X_\varepsilon^{t,x}(s)|,|U_\varepsilon(s)|]\ge N\Big]<\rho,\quad \varepsilon\in(0,\varepsilon_0]. \tag{6.15}$$

Thanks to (6.15), we can find a sufficiently large N so that

$$\limsup_{\varepsilon\to 0}\mathbb{P}\Big[\Big|\int_s^T\tilde F_\varepsilon(\sigma)d\sigma\Big|\ge\delta,\ \sup_{s\in[t,T]}\max[|X_\varepsilon^{t,x}(s)|,|U_\varepsilon(s)|]\ge N\Big]<\frac{\rho}{3}. \tag{6.16}$$

Let M be a non-negative integer and $t_j:=s+j(T-s)/M$, $j=0,\dots,M$. Using the tightness of $\big(X_\varepsilon^{t,x}(s),U_\varepsilon(s)\big)_{t\le s\le T}$, we can choose a sufficiently large M_0 so that

$$\limsup_{\varepsilon\to 0}\sup_{M\ge M_0}\mathbb{P}\Big[\int_s^T\Big|\tilde F_\varepsilon(\sigma)-\tilde F_{M,\varepsilon}(\sigma)\Big|d\sigma\ge\delta\Big]<\frac{\rho}{3}, \tag{6.17}$$

where

$$\tilde F_{M,\varepsilon}(s):=\tilde f\big(t_j,X_\varepsilon^{t,x}(t_j),U_\varepsilon(t_j),\eta_\varepsilon^{t,x}(s)\big),\quad t_j\le s<t_{j+1},\ j=0,\dots,M-1.$$

To prove (6.14), it suffices to show that given M, N and $\rho>0$ we have

$$\limsup_{\varepsilon\to 0}\mathbb{P}\Big[\Big|\int_s^T\tilde F_{M,\varepsilon}(\sigma)d\sigma\Big|\ge\delta,\ \sup_{s\in[t,T]}\max[|X_\varepsilon^{t,x}(s)|,|U_\varepsilon(s)|]\le N\Big]<\frac{\rho}{3}. \tag{6.18}$$

Obviously, we have

$$\Big|\int_s^T\tilde F_{M,\varepsilon}(\sigma)d\sigma\Big|\le\sum_{j=1}^{M-1}\Big|\int_{t_j}^{t_{j+1}}\tilde f\left(t_j,X_\varepsilon^{t,x}(t_j),U_\varepsilon(t_j),\eta_{\varepsilon,\sigma}^{t,x}\right)d\sigma\Big|.$$

Estimate (6.18) holds, provided we prove that for any $N>0$ and $t\le s\le s'\le T$:

$$\limsup_{\varepsilon\to 0}\mathbb{E}\Big[\sup_{(y,u)\in K_N}\Big|\int_s^{s'}\tilde f\left(s,y,u,\eta_{\varepsilon,\sigma}^{t,x}\right)d\sigma\Big|\Big]=0. \tag{6.19}$$

The latter however is a direct consequence of the decomposition (6.2) and Lemma 6.1. $\square$

Given $X\in C([t,T];\mathbb{R}^d)$ we let $U:=\Phi(X)\in C[t,T]$ be the unique solution of

$$u_0(X(T))=U(s)+\int_s^T\bar f(\sigma,X(\sigma),U(\sigma))d\sigma,\quad s\in[t,T]. \tag{6.20}$$

Suppose that $\mathfrak{Q}_\varepsilon$ are the laws of $\big(X_\varepsilon^{t,x}(s),U_\varepsilon(s)\big)_{s\le T}$ over $C([t,T];\mathbb{R}^{d+1})$ for $\varepsilon\in(0,1]$. Let $\mathfrak{Q}_*$ be the limiting law of $\mathfrak{Q}_{\varepsilon_n}$ for some sequence $\varepsilon_n\to 0$. Thanks to Proposition 6.2, we know that $\mathfrak{Q}_*$ is supported on the set

$$\mathcal{C}:=\{(X,U):X\in C([t,T];\mathbb{R}^d),\ U=\Phi(X)\}.$$

As $\big(X_\varepsilon^{t,x}(s)\big)_{s\in[t,T]}$ converges in law to $(x+\beta_{s-t})_{s\in[t,T]}$, as $\varepsilon\to 0$, and $u_\varepsilon(t,x)=U_\varepsilon(t)$, we know that for fixed (t,x), $u_\varepsilon(t,x)$ converges in distribution to $\mathcal{U}(t;t,x)$ with $\mathcal{U}(s;t,x)$ solving

$$u_0\left(x+\beta_{T-t}\right)-\mathcal{U}(s;t,x)=\int_s^T\bar f\left(\sigma,x+\beta_{\sigma-t},\mathcal{U}(\sigma;t,x)\right)d\sigma,\quad t\le s\le T. \tag{6.21}$$

Concerning the convergence of the multi-point statistics the argument from Section 4.4 proves that for N distinct points $x_1, \ldots, x_N \in \mathbb{R}^d$ the process $(X_\varepsilon^{t,x_1}(s), \ldots, X_\varepsilon^{t,x_N}(s))_{s\geq t}$ converges in law to $(x_1 + \beta_{s-t}^{(1)}, \ldots, x_1 + \beta_{s-t}^{(N)})_{s\geq t}$, where $(\beta_s^{(j)})_{s\geq 0}$, $j = 1, \ldots, N$ are i.i.d. Brownian motions with the covariance (4.10). This, in turn implies that $(\mathcal{U}^{(1)}(t, x_1), \ldots, \mathcal{U}^{(N)}(t, x_n))$, the respective limit of $(u_\varepsilon(t, x_1), \ldots, u_\varepsilon(t, x_N))$ is determined by the solutions of (6.21) based on $(\beta_s^{(j)})_{s\geq 0}$, $j = 1, \ldots, N$, thus they are independent. This ends the proof of Theorem 2.3. □

The proof of Corollary 2.4 follows essentially from the same argument as Corollary 2.2. It suffices only to note that from (6.12) it follows that $\|u_\varepsilon(t, \cdot)\|_{L^\infty(\mathbb{R}^d)}$, is deterministically bounded for $\varepsilon \in (0, 1)$. Thus the random variables $\langle u_\varepsilon(t), \varphi\rangle$ are also deterministically bounded , for any test function $\varphi \in L^1(\mathbb{R}^d)$. We can repeat then the argument used to show (5.23) to conclude (2.15). □

7 Proof of Theorem 2.6

7.1 The limiting dynamics and the proof of Proposition 2.5

Let us first explain how the coefficients b and $\tilde{c}_j$ in (2.20) are defined. We set

$$b(s, y, u) := \langle \nabla_y \Theta(s, y, u, \cdot) \cdot v(\cdot)\rangle_\pi + \langle \partial_u \Theta(s, y, u, \cdot) f(s, y, u, \cdot)\rangle_\pi, \tag{7.1}$$

where $\Theta : D_T \times \mathcal{E} \to \mathbb{R}$ is the solution of (6.1), with $\tilde{f}$ in the right side replaced by f – recall that now we assume f has mean zero.

In order to define $\tilde{c}_j$, recall that the constant matrix $S = a^{1/2}$, with a_{ij} given by (4.10):

$$a_{ij} := \mathcal{E}_L(\chi_i, \chi_j), \quad i, j = 1, \ldots, d.$$

In particular, a is non-singular. Then $\tilde{c}_0(s, y, u) \geq 0$ and $\tilde{c}^T(s, y, u) = [\tilde{c}_1(s, y, u), \ldots, \tilde{c}_d(s, y, u)]$ are determined by

$$S\tilde{c} = c, \quad \tilde{c}_0^2(s, y, u) := c_0(s, y, u) - \sum_{j=1}^d \tilde{c}_j^2(s, y, u) \tag{7.2}$$

with $c^T = [c_1, \ldots, c_d]$ and c_0 given by

$$c_0(s, y, u) = \mathcal{E}_L(\Theta(s, y, u)), \qquad c_j(s, y, u) := \mathcal{E}_L(\Theta(s, y, u), \chi_j), \quad j = 1, \ldots, d. \tag{7.3}$$

We can write

$$\tilde{c}_j(s, y, u) = \sum_{k=1}^d S_{jk}^{-1} c_k = \mathcal{E}_L(\Theta(s, y, u), \tilde{\chi}_j),$$

with the functions

$$\tilde{\chi}_j := \sum_{k=1}^d S_{jk}^{-1} \chi_k, \quad j = 1, \ldots, d,$$

that are orthonormal with respect to the inner product $\mathcal{E}_L(\cdot, \cdot)$:

$$\mathcal{E}_L(\tilde{\chi}_j, \tilde{\chi}_m) = \sum_{k,p=1}^d S_{jk}^{-1} S_{mp}^{-1} \mathcal{E}_L(\chi_k, \chi_p) = \sum_{k,p=1}^d S_{jk}^{-1} a_{kp} S_{mp}^{-1} = \delta_{jm}.$$

Thus, we have

$$\sum_{j=1}^d \tilde{c}_j^2(s, y, u) = \sum_{j=1}^d \mathcal{E}_L^2(\Theta(s, y, u), \tilde{\chi}_j) \leq \mathcal{E}_L(\Theta(s, y, u)) = c_0(s, y, u).$$

Let now $(X^{t,x}(s), U^{t,x,u}(s))$, $s \le T$ be the solution of (2.20) with the above coefficients b and $\tilde{c}_j$, fix $(t,x) \in \mathbb{R}^{1+d}$ and $T > t$, and let

$$\xi^{t,x}_s(u) := (\mathfrak{s}^{t,x}_s)'(u) = \frac{\partial}{\partial u} U^{t,x,u}(s), \quad (s,u) \in [t,T] \times \mathbb{R}.$$

By the equation satisfied by $U^{t,x,u}(s)$, it is clear that for a fixed u, the process $\big(\xi^{t,x}_s(u)\big)_{t \le s \le T}$ is a semimartingale that satisfies

$$\xi^{t,x}_s(u) = 1 + \int_t^s \alpha(\sigma)\xi^{t,x}_\sigma(u) d\sigma + \sum_{j=0}^d \int_t^s \gamma_j(\sigma)\xi^{t,x}_\sigma(u) d\tilde{\beta}_j(\sigma), \tag{7.4}$$

with

$$\alpha(\sigma) := \frac{\partial b}{\partial u}(\sigma, X^{t,x}(\sigma), U^{t,x,u}(\sigma)), \quad \gamma_j(\sigma) := \frac{\partial \tilde{c}_j}{\partial u}(\sigma, X^{t,x}(\sigma), U^{t,x,u}(\sigma)).$$

The unique solution of (7.4) is given by $\xi^{t,x}_s(u) = \exp\{Z(s)\}$, $s \in [t,T]$, with

$$Z(s) := \int_t^s \Big\{\alpha(\sigma) - \frac{1}{2}\sum_{j=0}^d \gamma_j^2(\sigma)\Big\} d\sigma + \sum_{j=0}^d \int_t^s \gamma_j(\sigma) d\tilde{\beta}_j(\sigma). \tag{7.5}$$

Thus, $\xi^{t,x}_s(u) > 0$ a.s., and since for any $s \in [t,T]$ we have

$$\lim_{u \to \pm\infty} U^{t,x,u}(s) = \pm\infty, \quad \text{a.s.},$$

we conclude from the above that $\mathfrak{s}^{t,x}_s(u) = U^{t,x,u}(s)$, $u \in \mathbb{R}$ is a diffeomorphism. This ends the proof of Proposition 2.5. $\square$

7.2 The truncated dynamics and convergence of the forward process

Recall that we use the notation $\mathfrak{S}^{t,x}_\varepsilon(s,u) = U^{t,x,u}_\varepsilon(s)$, where $U^{t,x,u}_\varepsilon(s)$ is the solution of

$$U^{t,x,u}_\varepsilon(s) = u + \frac{1}{\varepsilon}\int_t^s f_\varepsilon(\sigma) d\sigma, \quad t \le s \le T, \tag{7.6}$$

where we used the simplified notation

$$f_\varepsilon(\sigma) = f\Big(\sigma, X^{t,x}_\varepsilon(\sigma), U^{t,x,u}_\varepsilon(\sigma), V\Big(\frac{\sigma}{\varepsilon^2}, \frac{X^{t,x}_\varepsilon(\sigma)}{\varepsilon} + \cdot\Big)\Big) = f\Big(\sigma, X^{t,x}_\varepsilon(\sigma), U^{t,x,u}_\varepsilon(\sigma), \eta^{t,x}_{\varepsilon,\sigma}\Big).$$

The mapping $\mathfrak{s}^{t,x}_{s,\varepsilon}(u) := \mathfrak{S}^{t,x}_\varepsilon(s,u)$, is a diffeomorphism of $\mathbb{R}$ onto itself for each (t,x) and $t \le s \le T$. Indeed, for each fixed $\varepsilon > 0$, the derivative process

$$\xi^{t,x}_{s,\varepsilon}(u) := (\mathfrak{s}^{t,x}_{s,\varepsilon})'(u) = \frac{\partial}{\partial u} U^{t,x,u}_\varepsilon(s)$$

satisfies the linear equation

$$\xi^{t,x}_{s,\varepsilon}(u) = 1 + \frac{1}{\varepsilon}\int_t^s \frac{\partial f_\varepsilon}{\partial u}(\sigma)\xi^{t,x}_{\sigma,\varepsilon}(u) d\sigma, \tag{7.7}$$

thus $\xi^{t,x}_{s,\varepsilon}(u) > 0$ a.s., and since $\lim_{u \to \pm\infty} \mathfrak{s}^{t,x}_{s,\varepsilon}(u) = \pm\infty$ a.s., it is a diffeomorphism. Therefore, for $u_\varepsilon(t,x)$ to satisfy (2.16), it is equivalent to

$$\mathfrak{s}^{t,x}_{T,\varepsilon}(u_\varepsilon(t,x)) = u_0(X^{t,x}_\varepsilon(T)). \tag{7.8}$$

The first step in the proof of Theorem 2.6 is to establish tightness of the family $(X_\varepsilon^{t,x}(\cdot),\mathfrak{S}_\varepsilon^{t,x}(\cdot))$. However, instead of proving this directly, we will first prove tightness for a truncated family of processes $(X_\varepsilon^{t,x}(\cdot),\mathfrak{S}_{\varepsilon,M}^{t,x}(\cdot))$ – note that only the $\mathfrak{S}_\varepsilon^{t,x}$ component is truncated – and identify the corresponding limit as $\varepsilon\to 0$. Then, using the properties of the limit process, we will show that "truncation does not matter", and get the limit for the original, un-truncated process. To this end, take $M>1$ and set

$$f^{(M)}(s,y,v,w):=\phi_M(y,v)f(s,y,v,w),\quad (s,y,v,w)\in D_T\times\mathcal{E}.$$

Here, $\phi_M:\mathbb{R}^{1+d}\to[0,1]$ is a smooth cut-off function such that

$$\phi_M\equiv 1 \text{ on } K_M:=[(y,v):\ |y|\le M,\ |v|\le M],$$

and ϕ_M is supported in K_{M+2}, with $\|\nabla\phi_M\|_\infty\le 1$. We define $U_{\varepsilon,M}^{t,x,u}(s)$ as the solution of a modified equation (6.13):

$$U_{\varepsilon,M}^{t,x,u}(s)=u+\frac{1}{\varepsilon}\int_t^s f_\varepsilon^{(M)}(\sigma)d\sigma,\quad t\le s\le T,\tag{7.9}$$

with

$$f_\varepsilon^{(M)}(\sigma):=f^{(M)}(\sigma,X_\varepsilon^{t,x}(\sigma),U_{\varepsilon,M}^{t,x,u}(\sigma),\eta_{t,x}^\varepsilon(\sigma)),$$

where we write

$$\eta_{t,x}^\varepsilon(\sigma)=\eta_{\varepsilon,\sigma}^{t,x}$$

to emphasize its dependence on σ as a process. We denote by $\mathfrak{S}_{\varepsilon,M}^{t,x}(s,u)$ and $\mathfrak{s}_{s,\varepsilon,M}^{t,x}(u)$ the random field and family of diffeomorphisms corresponding to $U_{\varepsilon,M}^{t,x,u}(s)$, and by $u_{\varepsilon,M}(t,x)$ the unique solution of

$$\mathfrak{s}_{T,\varepsilon,M}^{t,x}\left(u_{\varepsilon,M}(t,x)\right)=u_0(X_\varepsilon^{t,x}(T)).\tag{7.10}$$

To define the limit of the truncated dynamics, let $U_M^{t,x,u}(s)$ be the solution of the SDE

$$U_M^{t,x,u}(s)=u+\int_t^s b_M(\sigma,X^{t,x}(\sigma),U_M^{t,x,u}(\sigma))d\sigma+\sum_{j=0}^d\int_t^s\tilde{c}_{M,j}(\sigma,X^{t,x}(\sigma),U_M^{t,x,u}(\sigma))d\tilde{\beta}_j(\sigma),\tag{7.11}$$

with b_M and $\tilde{c}_M$ as in (7.1), (7.2) and (7.3) but with Θ and f replaced by Θ_M and $f^{(M)}$, respectively. Here, $\Theta_M(s,y,v,w):=\phi_M(y,v)\Theta(s,y,v,w)$ is the solution to the cell problem (6.1) with $f^{(M)}$ in the right side. This generates the random field $\{\mathfrak{S}_M^{t,x}(s,u)=U_M^{t,x,u}(s)\}_{(s,u)\in[t,T]\times\mathbb{R}}$ and the corresponding diffeomorphisms $\mathfrak{s}_{T,M}^{t,x}(u):=\mathfrak{S}_M^{t,x}(T,u)$. We let $u_M(t,x)$ be the unique solution of

$$\mathfrak{s}_{T,M}^{t,x}\left(u_M(t,x)\right)=u_0(X^{t,x}(T)).\tag{7.12}$$

We will call $(\mathfrak{S}_{\varepsilon,M}^{t,x}(\cdot),X_\varepsilon^{t,x}(\cdot))$ the "forward process", and the goal of this section is

Proposition 7.1. *Given $M>1$ and $(t,x)\in[0,T]\times\mathbb{R}^d$, the processes $(\mathfrak{S}_{\varepsilon,M}^{t,x}(\cdot),X_\varepsilon^{t,x}(\cdot))$ converge weakly, as $\varepsilon\to 0$, over $C([t,T]\times\mathbb{R})\times C([t,T])$, equipped with the standard Frechet metric, to $(\mathfrak{S}_M^{t,x}(\cdot),X^{t,x}(\cdot))$.*

Proof. We will use the following notation: we set $g_{\varepsilon,M}(s,u):=g(s,X_\varepsilon^{t,x}(s),U_{\varepsilon,M}^{t,x,u}(s),\eta_{t,x}^\varepsilon(s))$ for a given field $g:D_T\times\mathcal{E}\to\mathbb{R}$, and also use $g'_{\varepsilon|u}(s)$ and $\nabla_x g_\varepsilon(s)$ to denote the processes corresponding to $g_u:=\partial g/\partial u$ and $\nabla_x g$, respectively. Using the Itô formula for

$$\Theta_\varepsilon(s)=\Theta(s,X_\varepsilon^{t,x}(s),U_\varepsilon^{t,x,u}(s),\eta_{t,x}^\varepsilon(s)),$$

and recalling that

$$-\mathcal{L}\Theta(t,x,u,w) = f(t,x,u,w),$$

we obtain

$$d\Theta_\varepsilon(s) = \Big\{\partial_s\Theta_\varepsilon(s) + \frac{1}{\varepsilon}\Big[\nabla_x\Theta_\varepsilon(s)\cdot v(\eta^\varepsilon_{t,x}(s)) + \Theta'_{\varepsilon|u}(s)\,f_\varepsilon(s)\Big] - \frac{1}{\varepsilon^2}f_\varepsilon(s)\Big\}\,ds \qquad (7.13)$$
$$+\frac{1}{\varepsilon}\left\langle \tau_{-X^{t,x}_\varepsilon(s)/\varepsilon}\mathcal{D}\Theta_\varepsilon(s), dB^\varepsilon_s\right\rangle_H,$$

which, in turn, gives

$$\frac{1}{\varepsilon}f_\varepsilon(s)\,ds = -\varepsilon d\Theta_\varepsilon(s) + \Big\{\varepsilon\partial_s\Theta_\varepsilon(s) + \nabla_x\Theta_\varepsilon(s)\cdot v(\eta^\varepsilon_{t,x}(s)) + \Theta'_{\varepsilon|u}(s)\,f_\varepsilon(s)\Big\}\,ds \qquad (7.14)$$
$$+\left\langle \tau_{-X^{t,x}_\varepsilon(s)/\varepsilon}\mathcal{D}\Theta_\varepsilon(s), dB^\varepsilon_s\right\rangle_H.$$

The obvious analogs of (7.13) and (7.14) for $\Theta_{\varepsilon,M}(s)$ and $f^{(M)}_\varepsilon(s)$, together with (7.9), lead to a decomposition

$$\mathfrak{S}^{t,x}_{\varepsilon,M}(s,u) = \sum_{j=0}^{2}\mathfrak{S}^{t,x}_{\varepsilon,M,j}(s,u), \qquad (7.15)$$

with

$$\mathfrak{S}^{t,x}_{\varepsilon,M,0}(s,u) := \varepsilon\Theta_{\varepsilon,M}(t,u) - \varepsilon\Theta_{\varepsilon,M}(s,u) + \varepsilon\int_t^s \partial_\sigma\Theta_{\varepsilon,M}(\sigma,u)\,d\sigma,$$
$$\mathfrak{S}^{t,x}_{\varepsilon,M,1}(s,u) := u + \int_t^s \Big\{\nabla_x\Theta_{\varepsilon,M}(\sigma,u)\cdot v(\eta^\varepsilon_{t,x}(\sigma)) + \Theta'_{\varepsilon,M|u}(\sigma,u)\,f_{\varepsilon,M}(\sigma,u)\Big\}d\sigma, \qquad (7.16)$$
$$\mathfrak{S}^{t,x}_{\varepsilon,M,2}(s,u) := \int_t^s \left\langle \tau_{-X^{t,x}_\varepsilon(\sigma)/\varepsilon}\mathcal{D}\Theta_{\varepsilon,M}(\sigma,u), dB^\varepsilon_\sigma\right\rangle_H, \quad t\le s\le T.$$

The terms in the right side of (7.15) satisfy several estimates given by Lemmas 7.2 and 7.4 below, which we will use to prove the tightness of $\mathfrak{S}^{t,x}_{\varepsilon,M}(s,u)$. Take arbitrary $\eta,\eta'>0$ and $N>1$. Since the right side of (7.16) vanishes for $|u|>M+2$, we may assume that $N\le M+2$. According to Corollary 7.3 and Lemma 7.4, we can choose $\delta>0$ and $L>1$ such that $\limsup_{\varepsilon\to0}\mathbb{P}[E_{\varepsilon,L,\delta,\eta'}]<\eta$, where

$$E_{\varepsilon,L,\delta,\eta'} := \Big[\sup_{\substack{t\le s<s'\le T,|u|\le N\\ s'-s<\delta}}\Big|\mathfrak{S}^{t,x}_{\varepsilon,M}(s',u)-\mathfrak{S}^{t,x}_{\varepsilon,M}(s,u)\Big| \ge \frac{\eta'}{2},\ \text{ or }\ \sup_{s\in[t,T],|u|\le N}|\xi^{t,x}_{s,\varepsilon,M}(u)|>L\Big] \qquad (7.17)$$

However, we have $E_{\varepsilon,L,\delta,\eta'}\supset E_{\varepsilon,\delta',\eta'}$, with $\delta':=\min[\delta,\eta'(2L)^{-1}]$, and

$$E_{\varepsilon,\delta',\eta'} := \Big[\sup_{\substack{s,s'\in[0,T],|u|,|u'|\le N\\ |s'-s|+|u-u'|<\delta'}}\Big|\mathfrak{S}^{t,x}_{\varepsilon,M}(s',u')-\mathfrak{S}^{t,x}_{\varepsilon,M}(s,u)\Big|\ge\eta'\Big]. \qquad (7.18)$$

Hence, $\limsup_{\varepsilon\to0}\mathbb{P}[E_{\varepsilon,\delta',\eta'}]<\eta$ and tightness follows from Theorem 2.7.3, p. 82 of [3].

In order to identify the limit, it suffices to prove that for any $n\ge1$ and $(s_1',u_1),\dots,(s_n',u_n)\in[t,T]\times\mathbb{R}$, $s_1,\dots,s_n\in[t,T]$, we have

$$\Big(\mathfrak{S}^{t,x}_{\varepsilon,M}(s_1',u_1),\dots,\mathfrak{S}^{t,x}_{\varepsilon,M}(s_n',u_n),X^{t,x}_\varepsilon(s_1),\dots,X^{t,x}_\varepsilon(s_n)\Big)$$
$$\overset{\varepsilon\to0}{\Longrightarrow}\Big(\mathfrak{S}^{t,x}_{M}(s_1',u_1),\dots,\mathfrak{S}^{t,x}_{M}(s_n',u_n),X^{t,x}(s_1),\dots,X^{t,x}(s_n)\Big). \qquad (7.19)$$

To show (7.19), we can use (5.1) together with (7.15) (recall that $\mathfrak{S}^{t,x}_{\varepsilon,M}(u) = U^{t,x,u}_{\varepsilon,M}(s)$) and apply a weak convergence argument for semimartingales analogous to the one used in Section 6. Since the argument is rather similar, we do not present the details. This finishes the proof of Proposition 7.1. $\square$

In the following, we present the technical lemmas used in the proof of Proposition 7.1.

Lemma 7.2. *For each $M > 1$ we have*

$$\lim_{\varepsilon\to 0} \mathbb{E}\Big[\sup_{s\in[t,T],|u|\le M+2} \Big|\mathfrak{S}^{t,x}_{\varepsilon,M,0}(s,u)\Big|\Big] = 0. \tag{7.20}$$

In addition, we have

$$\lim_{\delta\to 0}\limsup_{\varepsilon\to 0} \mathbb{E}\Big[\sup_{\substack{t\le s<s'\le T,|u|\le M+2\\ s'-s<\delta}} \Big|\mathfrak{S}^{t,x}_{\varepsilon,M,1}(s',u) - \mathfrak{S}^{t,x}_{\varepsilon,M,1}(s,u)\Big|\Big] = 0, \tag{7.21}$$

and, for any $\eta > 0$ we have

$$\lim_{\delta\to 0}\limsup_{\varepsilon\to 0} \mathbb{P}\Big[\sup_{\substack{t\le s<s'\le T,|u|\le M+2\\ s'-s<\delta}} \Big|\mathfrak{S}^{t,x}_{\varepsilon,M,2}(s',u) - \mathfrak{S}^{t,x}_{\varepsilon,M,2}(s,u)\Big| > \eta\Big] = 0. \tag{7.22}$$

As a direct corollary, we conclude the following.

Corollary 7.3. *For each $M > 1$ and $\eta > 0$, we have*

$$\lim_{\delta\to 0}\limsup_{\varepsilon\to 0} \mathbb{P}\Big[\sup_{\substack{t\le s<s'\le T,|u|\le M+2\\ s'-s<\delta}} \Big|\mathfrak{S}^{t,x}_{\varepsilon,M}(s',u) - \mathfrak{S}^{t,x}_{\varepsilon,M}(s,u)\Big| > \eta\Big] = 0. \tag{7.23}$$

We will also need a bound on the derivative process

$$\xi^{t,x}_{s,\varepsilon,M}(u) := (\mathfrak{s}^{t,x}_{s,\varepsilon,M})'(u) = \frac{\partial}{\partial u} U^{t,x,u}_{\varepsilon,M}(s),$$

which satisfies an integral equation

$$\xi^{t,x}_{s,\varepsilon,M}(u) = 1 + \frac{1}{\varepsilon}\int_t^s f'_{\varepsilon,M|u}(\sigma)\xi^{t,x}_{\sigma,\varepsilon,M}(u)d\sigma. \tag{7.24}$$

Lemma 7.4. *For any $M > 1$, we have*

$$\lim_{L\to+\infty}\limsup_{\varepsilon\to 0} \mathbb{P}\Big[\sup_{s\in[t,T],|u|\le M+2} |\xi^{t,x}_{s,\varepsilon,M}(u)| > L\Big] = 0. \tag{7.25}$$

Proof of Lemma 7.2

Proof of (7.20) and (7.21). Using the Sobolev embedding, we can estimate

$$\sup_{s\in[t,T],|u|\le M+2} |\Theta_{\varepsilon,M}(s,u)| \le \sup_{(s,y,v)\in\tilde D_{M+2}} |\Theta(s,y,v)| \le C_M\left\{\|\Theta\|_{L^p(\tilde D_{M+2})} + \|\nabla\Theta\|_{L^p(\tilde D_{M+2})}\right\}, \tag{7.26}$$

with $p > d+2$, constant $C_M > 0$ independent of ε and

$$\tilde D_M := [t,T]\times K_M.$$

Taking the expectation in both sides of (7.26), we obtain that, for each $N, M \ge 1$:

$$\lim_{\varepsilon\to 0} \mathbb{E}\Big[\varepsilon \sup_{s\in[t,T],\,|u|\le M+2} |\Theta_{\varepsilon,M}(s,u)|\Big] = 0, \tag{7.27}$$

and (7.20) follows.

A similar argument shows that

$$\begin{aligned} &\limsup_{\varepsilon\to 0} \mathbb{E}\Big[\sup_{s\in[t,T],\,|u|\le M+2} |\nabla_x \Theta_{\varepsilon,M}(s,u)\cdot v(\eta^{\varepsilon}_{t,x}(s))|\Big] < +\infty, \\ &\limsup_{\varepsilon\to 0} \mathbb{E}\Big[\sup_{s\in[t,T],\,|u|\le M+2} |\Theta'_{\varepsilon,M|u}(s,u) f_{\varepsilon,M}(s,u)|\Big] < +\infty. \end{aligned} \tag{7.28}$$

These estimates imply (7.21).

Proof of (7.22). We start with the following "finite-rank" approximation.

Lemma 7.5. *Let $M > 1$, $m, m_1 \ge 0$ and $f : \tilde{D}_M \times \mathcal{E} \to \mathbb{R}$ be such that*

$$\max_{|k|\le m_1} \operatorname{esssup}_{w\in\mathcal{E}} \|D^k f\|_{C^m(\tilde{D}_M)} < +\infty.$$

Then, for any $\delta > 0$, there exist $\varphi_1, \ldots, \varphi_N \in C^m(D_T)$ and $\Phi_1, \ldots, \Phi_N \in \mathcal{W}_{m_1,\infty}$ (cf (3.6)) such that

$$\max_{|k|\le m_1} \operatorname{esssup}_{w\in\mathcal{E}} \|D^k f(\cdot,w) - D^k \tilde{f}(\cdot,w)\|_{C^m(\tilde{D}_M)} < \delta, \tag{7.29}$$

where

$$\tilde{f}(s,y,u,w) := \sum_{j=1}^{N} \varphi_j(s,y,u)\Phi_j(w), \quad (s,y,u,w) \in D_T \times \mathcal{E}. \tag{7.30}$$

Proof. To simplify the presentation, we assume that f does not depend on y and $k = 0$. Let us partition the rectangle $[t,T]\times[-M,M]$ using the grid points

$$t = t_0 < \ldots < t_n = T,\ -M = m_0 < \ldots < m_n = M,\ t_i - t_{i-1} = \frac{T-t}{n},\ m_i - m_{i-1} = \frac{2M}{n}, i = 1, \ldots, n.$$

Let $\Delta_{i,i'} := [t_{i-1}, t_i] \times [m_{i'-1}, m_{i'}]$ and

$$m^*_{i'} := \frac{m_{i'} + m_{i'-1}}{2}, \quad t^*_i := \frac{t_i + t_{i-1}}{2}, \quad i, i' = 1, \ldots, n,$$

and $\Delta^0_{i,i'}$ be an open neighborhood of $\Delta_{i,i'}$ contained in

$$\left[t_{i-1} - \frac{T-t}{2n}, t_i + \frac{T-t}{2n}\right] \times \left[m_{i'-1} - \frac{M}{n}, m_{i'} + \frac{M}{n}\right].$$

Let $\phi_{i,i'} : \mathbb{R}^2 \to [0,1]$ be a smooth partition of unity on $[t,T]\times[-M,M]$ subordinated to the open covering $\left(\Delta^0_{i,i'}\right)$ of $[t,T]\times[-M,M]$. We may assume that $\phi_{i,i'} \equiv 1$ in some neighborhood of $(t^*_i, m^*_{i'})$.

Since the function $(s,u) \mapsto f(s,u,\cdot)$ is uniformly continuous from $[t,T]\times[-M,M]$ to $L^\infty(\pi)$, we can choose n sufficiently large so that $|f(s,u) - f(s',u')| < \delta$ for $(s,u),(s',u') \in \Delta^0_{i,i'}$. Let

$$\tilde{f}(s,y,u,w) := \sum_{i,i'} \phi_{i,i'}(s,u) f(t^*_i, m^*_{i'}, w), \quad (s,u,w) \in [t,T]\times[-M,M]\times\mathcal{E}.$$

One can easily verify that then (7.29) holds with $m = 0$. $\square$

We go back to the proof of Lemma 7.2. Choose arbitrary $\delta, \eta > 0$ and choose $\tilde{f}$, of the form (7.30), so that it satisfies (7.29), with δ, f replaced by $\delta\eta$ and f_M, respectively, and $m = d+2$. Let Θ_j be the solution of

$$-\mathcal{L}\Theta_j = \Phi_j,$$

with Φ_j constructed in Lemma 7.5, then $\tilde{\Theta}^{(0)}(s,y,u,w) := \sum_{j=1}^N \varphi_j(s,y,u)\Theta_j(w)$ satisfies

$$\sum_{\ell=0}^{d+2} \sum_{|k|=\ell} \sup_{(s,y,u)\in\tilde{D}_{M+2}} \mathcal{E}_L\left(\nabla^k\tilde{\Theta}^{(0)}(s,y,u) - \nabla^k\Theta_M(s,y,u)\right) \le C\frac{(\delta\eta)^2}{\alpha_*}.$$

The constant C depends only on d. Thus, approximating Θ_j, if needed, we can find $\tilde{\Theta}_j \in L^\infty(\pi)$ such that $\mathcal{D}\tilde{\Theta}_j \in L^\infty(\pi, H)$, and

$$\sum_{\ell=0}^{d+2} \sum_{|k|=\ell} \sup_{(s,y,u)\in\tilde{D}_{M+2}} \mathcal{E}_L(\nabla^k\tilde{\Theta}(s,y,u) - \nabla^k\Theta_M(s,y,u)) \le C\frac{(\delta\eta)^2}{\alpha_*}, \tag{7.31}$$

with

$$\tilde{\Theta}(s,y,u,w) := \sum_{j=1}^N \varphi_j(s,y,u)\tilde{\Theta}_j(w). \tag{7.32}$$

Define

$$\tilde{\mathfrak{S}}_\varepsilon^{t,x}(s,u) := \int_t^s \langle \tau_{-X_\varepsilon^{t,x}(\sigma)/\varepsilon}\mathcal{D}\tilde{\Theta}_\varepsilon\,(\sigma,u)\,, dB_\sigma^\varepsilon\rangle_H, \tag{7.33}$$

where

$$\tilde{\Theta}_\varepsilon(s,u) = \tilde{\Theta}(s, X_\varepsilon^{t,x}(s), U_\varepsilon^{t,x,u}(s), \eta_{t,x}^\varepsilon(s)).$$

By the Sobolev embedding, there exists a constant $C > 0$ such that

$$\begin{aligned} &\sup_{s\in[t,T],|u|\le M+2} |\tilde{\mathfrak{S}}_\varepsilon^{t,x}(s,u) - \mathfrak{S}_{\varepsilon,M,2}^{t,x}(s,u)| \\ &\le C\int_{-M-2}^{M+2}\Big\{\sup_{s\in[t,T]}|\tilde{\mathfrak{S}}_\varepsilon^{t,x}(s,u) - \mathfrak{S}_{\varepsilon,M,2}^{t,x}(s,u)| + \sup_{s\in[t,T]}|\partial_u\tilde{\mathfrak{S}}_\varepsilon^{t,x}(s,u) - \partial_u\mathfrak{S}_{\varepsilon,M,2}^{t,x}(s,u)|\Big\}du. \end{aligned} \tag{7.34}$$

Applying expectation to both sides of (7.34) and using Doob's inequality, we obtain

$$\begin{aligned} &\mathbb{E}\Big[\sup_{s\in[t,T],|u|\le M+2}\Big|\tilde{\mathfrak{S}}_\varepsilon^{t,x}(s,u) - \mathfrak{S}_{\varepsilon,M,2}^{t,x}(s,u)\Big|\Big] \\ &\le C\int_{-M-2}^{M+2} du\,\mathbb{E}\Big\{\int_t^T\Big[\Big|\Big(\mathcal{D}\tilde{\Theta}_\varepsilon - \mathcal{D}\Theta_{\varepsilon,M}\Big)(s,u)\Big|_H^2 + \Big|\Big(\partial_u\mathcal{D}\tilde{\Theta}_\varepsilon - \partial_u\mathcal{D}\Theta_{\varepsilon,M}\Big)(s,u)\Big|_H^2\Big]ds\Big\}^{1/2} \\ &\le 2C(M+2)\mathbb{E}\Big\{\int_t^T\sup_{(y,v)\in K_{M+2}}\Big[\Big|(\mathcal{D}\tilde{\Theta} - \mathcal{D}\Theta_M)(s,y,v,\eta_{t,x}^\varepsilon(s))\Big|_H^2 \\ &+ \Big|\Big(\partial_v\mathcal{D}\tilde{\Theta} - \partial_v\mathcal{D}\Theta_M\Big)(s,y,v,\eta_{t,x}^\varepsilon(s))\Big|_H^2\Big]ds\Big\}^{1/2}. \end{aligned} \tag{7.35}$$

Using again the Sobolev estimate, this time to estimate the supremum of

$$\sup_{(y,v)\in K_{M+2}}|(\mathcal{D}\tilde{\Theta} - \mathcal{D}\Theta_M)(s,y,v,\eta_{t,x}^\varepsilon(s))|_H^2,$$

we conclude that there exist constants $C, C' > 0$ such that

$$\sup_{(y,v)\in K_{M+2}}|(\mathcal{D}\tilde{\Theta} - \mathcal{D}\Theta_M)(s,y,v,\eta_{t,x}^\varepsilon(s))|_H^2 \le C'\sum_{k=0}^{d+1}\int_{K_{M+2}}|\nabla^k(\mathcal{D}\tilde{\Theta} - \mathcal{D}\Theta_M)(s,y,v,\eta_{t,x}^\varepsilon(s))|_H^2 dydv.$$

A similar estimate holds for $|\partial_v(\mathcal{D}\tilde{\Theta} - \mathcal{D}\Theta_M)(s,y,v,\eta^\varepsilon_{t,x}(s))|^2_H$, leading to

$$
\begin{aligned}
&\mathbb{E}\Big[\sup_{s\in[t,T],|u|\le M+2}|\tilde{\mathfrak{S}}^{t,x}_\varepsilon(s,u) - \mathfrak{S}^{t,x}_{\varepsilon,M,2}(s,u)|\Big] \\
&\le 2C(M+2)\Big\{\sum_{\ell=0}^{d+1}\sum_{|k|=\ell}\int_{\mathcal{E}} d\pi\Big[\int_{\tilde{D}_{M+2}}|\nabla^k(\mathcal{D}\tilde{\Theta} - \mathcal{D}\Theta_M)(s,y,v)|^2_H ds dy dv\Big]\Big\}^{1/2},
\end{aligned}
\tag{7.36}
$$

with a constant $C > 0$ depending only on M and d. By virtue of (7.31), we conclude that

$$
\mathbb{E}\Big[\sup_{s\in[t,T],|u|\le M+2}|\tilde{\mathfrak{S}}^{t,x}_\varepsilon(s,u) - \mathfrak{S}^{t,x}_{\varepsilon,M,2}(s,u)|\Big] \le C\delta\eta\alpha_*^{-1/2}. \tag{7.37}
$$

It follows from the Chebyshev inequality that there exists $C > 0$ depending only on M, d and α_* such that for any $\delta, \eta > 0$ we can find $\tilde{\Theta}$ of the form (7.32) such that

$$
\limsup_{\varepsilon\to 0}\mathbb{P}\Big[\sup_{s\in[t,T],|u|\le M+2}|\tilde{\mathfrak{S}}^{t,x}_\varepsilon(s,u) - \mathfrak{S}^{t,x}_{\varepsilon,M,2}(s,u)| > \eta\Big] \le C\delta. \tag{7.38}
$$

The above, in particular, implies (7.21), if we prove that for any $\tilde{\mathfrak{S}}^{t,x}_\varepsilon(s,u)$ of the form (7.33) and any $\eta > 0$ we have

$$
\lim_{\delta\to 0}\limsup_{\varepsilon\to 0}\mathbb{P}[\tilde{Z}_{\eta,\delta,\varepsilon}] = 0, \tag{7.39}
$$

with

$$
\tilde{Z}_{\eta,\delta,\varepsilon} := \Big[\sup_{\substack{t\le s<s'\le T,|u|\le M+2\\ s'-s<\delta}}\Big|\tilde{\mathfrak{S}}^{t,x}_\varepsilon(s',u) - \tilde{\mathfrak{S}}^{t,x}_\varepsilon(s,u)\Big| > \eta\Big].
$$

To this end, we invoke the Sobolev inequality, as in (7.34) and (7.35). This, together with Burkolder-Davis-Gundy inequality, implies

$$
\mathbb{E}\Big[\sup_{|u|\le M+2}[\tilde{\mathfrak{S}}^{t,x}_\varepsilon(s',u) - \tilde{\mathfrak{S}}^{t,x}_\varepsilon(s,u)]^4\Big] \le C\int_{-M-2}^{M+2} du\,\mathbb{E}\Big\{\int_s^{s'}\Big(|\mathcal{D}\tilde{\Theta}_\varepsilon(\sigma,u)|^2_H + |\partial_u\mathcal{D}\tilde{\Theta}_\varepsilon(\sigma,u)|^2_H\Big)d\sigma\Big\}^2,
$$

for any $s' > s$. Since N is finite, we also have

$$
\operatorname{esssup}_{w\in\mathcal{E}}\sup_{(s,y,u)\in\tilde{D}_{M+2}}\big(|\mathcal{D}\tilde{\Theta}(s,y,u)|_H + |\partial_u\mathcal{D}\tilde{\Theta}(s,y,u)|_H\big) < +\infty, \tag{7.40}
$$

whence

$$
A_* := \mathbb{E}\Big[\int_t^T\int_t^T (s-s')^{-5/2}\sup_{|u|\le M+2}\big[\tilde{\mathfrak{S}}^{t,x}_\varepsilon(s',u) - \tilde{\mathfrak{S}}^{t,x}_\varepsilon(s,u)\big]^4 ds ds'\Big] < +\infty.
$$

Let $\rho > 0$ be arbitrary. By virtue of Chebyshev inequality we obtain that $\mathbb{P}[Z_{\rho,\varepsilon}] < \rho$ for all $\varepsilon \in (0,1]$, with

$$
Z_{\rho,\varepsilon} := \left[\int_t^T\int_t^T (s-s')^{-5/2}\sup_{|u|\le M+2}\Big[\tilde{\mathfrak{S}}^{t,x}_\varepsilon(s',u) - \tilde{\mathfrak{S}}^{t,x}_\varepsilon(s,u)\Big]^4 ds ds' > \frac{A_*}{\rho}\right].
$$

The Garcia-Rodemich-Rumsey estimate, see Theorem 2.1.3 of [32], implies that given $\rho > 0$ there exists $\delta_0 > 0$ such that

$$
\sup_{\substack{t\le s<s'\le T,|u|\le M+2\\ s'-s<\delta}}\Big|\tilde{\mathfrak{S}}^{t,x}_\varepsilon(s',u) - \tilde{\mathfrak{S}}^{t,x}_\varepsilon(s,u)\Big| \le 40\sqrt{2}\left(\frac{A_*}{\rho}\right)^{1/4}\delta^{1/8} < \eta, \quad \varepsilon\in(0,1],\ \delta\in(0,\delta_0)
$$

on the event $Z^c_{\rho.\varepsilon}$. Hence, for any $\rho > 0$ there exists $\delta_0 > 0$ such that $\tilde{Z}_{\eta,\delta,\varepsilon} \subset Z_{\rho,\varepsilon}$ for all $\varepsilon \in (0,1]$ and $\delta \in (0, \delta_0)$, which yields

$$\limsup_{\delta\to 0}\limsup_{\varepsilon\to 0}\mathbb{P}\left[\tilde{Z}_{\eta,\delta,\varepsilon}\right] \le \rho.$$

This in turn implies (7.39), as $\rho > 0$ can be made arbitrarily small. This ends the proof of (7.22) and thus that of Lemma 7.2. □

Proof of Lemma 7.4

Arguing as in the proof of (7.21), we can show that

$$\limsup_{\varepsilon\to 0}\mathbb{E}\Big[\sup_{|u|\le M+2,\, s\in[t,T]}\{|\Theta'_{\varepsilon,M|u}(s,u)| + |\partial_s\Theta'_{\varepsilon,M|u}(s,u)| + |\nabla_x\Theta'_{\varepsilon,M|u}(s,u)|\}\Big] < +\infty. \tag{7.41}$$

Given $L > 1$, define the event

$$E_{L,\varepsilon} := \Big[\sup_{|u|\le M+2,\, s\in[t,T]}\Big\{|\Theta'_{\varepsilon,M|u}(s,u)| + |\partial_s\Theta'_{\varepsilon,M|u}(s,u)| + |\nabla_x\Theta'_{\varepsilon,M|u}(s,u)|\Big\} > L\Big].$$

It follows from (7.41) that for an arbitrary $\delta > 0$, there exists $L > 1$ so that

$$\mathbb{P}[E_{L,\varepsilon}] < \delta, \qquad \text{for all } \varepsilon \in (0, (2L)^{-1}). \tag{7.42}$$

Next, on the event $E^c_{L,\varepsilon}$, let us define

$$\tilde{\xi}^{t,x}_{s,\varepsilon,M}(u) := \Theta_{1,\varepsilon,M}(s)\xi^{t,x}_{s,\varepsilon,M}(u), \quad \Theta_{1,\varepsilon,M}(s) := 1 + \varepsilon\Theta'_{\varepsilon,M|u}(s).$$

Recall that

$$\xi^{t,x}_{s,\varepsilon,M}(u) = 1 + \frac{1}{\varepsilon}\int_t^s f'_{\varepsilon,M|u}(\sigma)\xi^{t,x}_{\sigma,\varepsilon,M}(u)d\sigma,$$

using the Itô formula and the fact that $-\mathcal{L}\Theta'_{M|u} = f'_{M|u}$, we can write

$$\tilde{\xi}^{t,x}_{s,\varepsilon,M}(u) = 1 + \varepsilon\Theta'_{\varepsilon,M|u}(t,x,u) + \int_t^s \tilde{\xi}^{t,x}_{\sigma,\varepsilon,M}(u)\tilde{\gamma}_\varepsilon(d\sigma) + \int_t^s \tilde{\xi}^{t,x}_{\sigma,\varepsilon,M}(u)\tilde{\alpha}_\varepsilon(\sigma)d\sigma, \tag{7.43}$$

with

$$\tilde{\gamma}_\varepsilon(d\sigma) := \Theta^{-1}_{1,\varepsilon,M}(\sigma)\langle\tau_{-X^{t,x}_\varepsilon(\sigma)/\varepsilon}\mathcal{D}\Theta'_{\varepsilon,M|u}(\sigma), dB^\varepsilon_\sigma\rangle_H,$$

$$\tilde{\alpha}_\varepsilon(\sigma) := \Theta^{-1}_{1,\varepsilon,M}(\sigma)\Big\{\varepsilon\partial_\sigma\Theta'_{\varepsilon,M|u}(\sigma) + \nabla_x\Theta'_{\varepsilon,M|u}(\sigma)\cdot v(\eta^\varepsilon_{t,x}(\sigma)) + \Theta''_{\varepsilon,M|u}(\sigma)f_{\varepsilon,M}(\sigma) + \Theta'_{\varepsilon,M|u}(\sigma)f'_{\varepsilon,M|u}(\sigma)\Big\}.$$

The above allow us to write

$$\tilde{\xi}^{t,x}_{s,\varepsilon,M}(u) = (1 + \varepsilon\Theta'_{\varepsilon,M|u}(t,x,u))\exp\{\tilde{\mathcal{Z}}_\varepsilon(s,u)\}, \tag{7.44}$$

with

$$\tilde{\mathcal{Z}}_\varepsilon(s,u) := \int_t^s\{\tilde{\alpha}_\varepsilon(\sigma) - \frac{1}{2}\langle\tilde{\gamma}_\varepsilon\rangle_\sigma\}d\sigma + \int_t^s\tilde{\gamma}_\varepsilon(d\sigma), \tag{7.45}$$

and

$$\langle\tilde{\gamma}_\varepsilon\rangle_\sigma := \Theta^{-2}_{1,\varepsilon,M}(\sigma)\langle A\mathcal{D}\Theta'_{\varepsilon,M|u}(\sigma), \mathcal{D}\Theta'_{\varepsilon,M|u}(\sigma)\rangle_H.$$

Using the Sobolev embedding argument, as in (7.35) – (7.36), we see that

$$\limsup_{\varepsilon\to 0}\int_t^T\mathbb{E}\Big[\sup_{|u|\le M+2}\langle A\mathcal{D}\Theta'_{\varepsilon,M|u}(\sigma), \mathcal{D}\Theta'_{\varepsilon,M|u}(\sigma)\rangle_H\Big]d\sigma < +\infty. \tag{7.46}$$

Combining (7.41) and (7.46), we conclude that for any $\delta > 0$ there exists $L > 1$ such that

$$\limsup_{\varepsilon\to 0}\mathbb{P}\Big[\sup_{|u|\le M+2,\, s\in[t,T]}|\tilde{\mathcal{Z}}_\varepsilon(s,u)| > L,\, E^c_{L,\varepsilon}\Big] < \delta.$$

This together with (7.42) implies (7.25). The proof of Lemma 7.4 is complete. □

7.3 The weak convergence of $u_{\varepsilon,M}(t,x)$

The goal in this section is to show that not only the "forward" processes $(\mathfrak{S}^{t,x}_{\varepsilon,M}(\cdot), X^{t,x}_{\varepsilon}(\cdot))$ converge as $\varepsilon \to 0$ but also the "inverse" processes, i.e., $u_{\varepsilon,M}(t,x)$ and $u_M(t,x)$ given by (7.10) and (7.12), are close in law. Given $t \in [0,T]$, define

$$\mathfrak{X} := C([t,T] \times \mathbb{R}; \mathbb{R}) \times C([t,T]; \mathbb{R}^d).$$

Proposition 7.6. *The random elements $(\mathfrak{S}^{t,x}_{\varepsilon,M}(\cdot), X^{t,x}_{\varepsilon}(\cdot), u_{\varepsilon,M}(t,x))$ converge, as $\varepsilon \to 0$, in law over $C([t,T] \times \mathbb{R}; \mathbb{R}) \times C([t,T]; \mathbb{R}^d) \times \mathbb{R}$, to $(\mathfrak{S}^{t,x}_{M}(\cdot), X^{t,x}(\cdot), u_M(t,x))$.*

Proof. Given any $N > 1$, denote by $\mathcal{CM}_+(N)$ the G_δ subset of $\mathfrak{X}$ that consists of all $(S(\cdot), x(\cdot)) \in \mathfrak{X}$ such that S is a continuous function $S : [t,T] \times \mathbb{R} \to \mathbb{R}$, strictly increasing in the second variable, and $S(s,u) \equiv u$ for all $s \in [t,T]$, $|u| \geq N$. Also, for a $\mathfrak{X}$-valued random element $(\mathfrak{S}, X)$, denote by $\mathcal{L}(\mathfrak{S}, X)$ its law.

Recall that u_0, the terminal condition of (2.16), belongs to $C_0^\infty(\mathbb{R}^d)$, so there exists $K > 1$ such that the range of $u_0(\cdot)$ is contained in $[-K, K]$. Then, we have

$$\mathfrak{s}^{t,x}_{T,\varepsilon,M}(u) \equiv u, \text{ for } |u| \geq M_* := \max[M+2, K],$$

and, the laws $\mathcal{L}_\varepsilon := \mathcal{L}(\mathfrak{S}^{t,x}_{\varepsilon,M}(\cdot), X^{t,x}_{\varepsilon}(\cdot))$ and $\mathcal{L} := \mathcal{L}(\mathfrak{S}^{t,x}_{M}(\cdot), X^{t,x}(\cdot))$ are supported in $\mathcal{CM}_+(M_*)$.

Let $\mathcal{H} : \mathfrak{X} \to \mathbb{R}$ be given by

$$\mathcal{H}(S, X) := S^{-1}(T, u_0(X(T))), \quad (S, X) \in \mathcal{CM}_+(M_*).$$

Here $S^{-1}(T, \cdot)$ is the inverse of S in the second variable. Outside of $\mathcal{CM}_+(M_*)$, we can define $\mathcal{H}$ arbitrarily, for instance, as a constant. The mapping is measurable, bounded and continuous on $\mathcal{CM}_+(M_*)$.

We know from Proposition 7.1 that $(\mathfrak{S}^{t,x}_{\varepsilon_n,M}(\cdot), X^{t,x}_{\varepsilon_n}(\cdot))$ converge in law to $(\mathfrak{S}^{t,x}_{M}(\cdot), X^{t,x}(\cdot))$, for any $\varepsilon_n \to 0$. We also have

$$u_{\varepsilon_n,M}(t,x) = \mathcal{H}(\mathfrak{S}^{t,x}_{\varepsilon_n,M}(\cdot), X^{t,x}_{\varepsilon_n}(\cdot)) \quad \text{and} \quad u_M(t,x) = \mathcal{H}(\mathfrak{S}^{t,x}_{M}(\cdot), X^{t,x}(\cdot)). \tag{7.47}$$

By the continuous mapping theorem, see Theorem 2.7, p. 21 of [3], $u_{\varepsilon_n,M}(t,x)$ converges in law to $u_M(t,x)$, as $n \to +\infty$. $\square$

7.4 Proof of part (i) of Theorem 2.6

Let us denote by $\mathfrak{Q}_\varepsilon$, $\mathfrak{Q}_{\varepsilon,M}$, $\mathfrak{Q}_M$, $\mathfrak{Q}$ the respective laws of the random elements

$$\mathfrak{Y}_\varepsilon := (\mathfrak{S}^{t,x}_{\varepsilon}(\cdot), X^{t,x}_{\varepsilon}(\cdot)),\ \mathfrak{Y}_{\varepsilon,M} := (\mathfrak{S}^{t,x}_{\varepsilon,M}(\cdot), X^{t,x}_{\varepsilon}(\cdot)),\ \mathfrak{Y}_M := (\mathfrak{S}^{t,x}_{M}(\cdot), X^{t,x}(\cdot)),\ \mathfrak{Y} := (\mathfrak{S}^{t,x}(\cdot), X^{t,x}(\cdot)),$$

over $\mathfrak{X}$. By Proposition 7.1, we have $\mathfrak{Q}_{\varepsilon,M} \Longrightarrow \mathfrak{Q}_M$, as $\varepsilon \to 0$ for each $M > 1$. A standard argument based on local uniform convergence of coefficients b_M to b and $\tilde{c}_M$ to $\tilde{c}$, see e.g. Section 9.2.6, pp. 528-529 of [13], implies that also $\mathfrak{Q}_M \Longrightarrow \mathfrak{Q}$, as $M \to +\infty$.

Given $N, M > 1$, recall that

$$K_M = \{(y,u) :\ |y| < M,\ |u| < M\},$$

we define $T_{M,N} : \mathfrak{X} \to [0, T+1]$ by

$$T_{M,N}(\mathfrak{S}, X) := \inf \{s \in [t,T] :\ (X(s), \mathfrak{S}(s,u)) \notin K_M \quad \text{for some } |u| \leq N\}.$$

We adopt the convention that $T_{M,N}(\mathfrak{S},X) := T+1$, if the set, over which the infimum is taken, is empty.

Let $(\mathcal{F}_{N,s})_{N>1,s\in[t,T]}$ be the family of σ-algebras generated by $(\mathfrak{S}(\sigma,u),X(\sigma))$, with $|u|\le N$, and $\sigma\in[t,s]$. Note that

$$\mathfrak{Q}_\varepsilon[A,T_{M,N}>T]=\mathfrak{Q}_{\varepsilon,M}[A,T_{M,N}>T] \quad \text{and} \quad \mathfrak{Q}_M[A,T_{M,N}>T]=\mathfrak{Q}[A,T_{M,N}>T]$$

for any $A\in\mathcal{F}_{N,T}$, $M,N>1$, $\varepsilon>0$.

The following estimate on the random time $T_{M,N}$ is crucial in removing the truncation.

Proposition 7.7. *For any $\rho>0$ and $N>1$, there exists $M>1$ such that*

$$\sup_{M'\ge M}\mathfrak{Q}[T_{M',N}\le T]<\rho. \tag{7.48}$$

Proof. Since $\xi_s^{t,x}(u)=\partial_u U^{t,x,u}(s)>0$, we have

$$U^{t,x,N}(s)\ge U^{t,x,u}(s)\ge U^{t,x,-N}(s)\text{ for }|u|\le N.$$

It suffices to show that for any $\rho>0$ and $N>1$, there exists $M>1$ such that

$$\mathbb{P}\Big[\sup_{s\in[t,T]}|X^{t,x}(s)|\ge M\Big]+\mathbb{P}\Big[\sup_{s\in[t,T]}|U^{t,x,N}(s)|\ge M\Big]+\mathbb{P}\Big[\sup_{s\in[t,T]}|U^{t,x,-N}(s)|\ge M\Big]<\rho. \tag{7.49}$$

Since $(X^{t,x}(s))_{s\in[t,T]}$ is a Brownian motion, there exist $C_1,C_2>0$ such that

$$\mathbb{P}\Big[\sup_{s\in[t,T]}|X^{t,x}(s)|\ge M\Big]\le C_1\exp\Big\{-C_2\frac{M^2}{T-t}\Big\}<\frac{\rho}{3},$$

provided that $M>1$ is sufficiently large. Similarly, we deduce from (2.20) that

$$\mathbb{P}\Big[\sup_{s\in[t,T]}|U^{t,x,N}(s)|\ge M\Big]\le C_1\exp\Big\{-C_2\frac{(M-N-\|b\|_\infty(T-t))^2}{T-t}\Big\}<\frac{\rho}{3},$$

for $M-N$ sufficiently large. A similar estimate holds for $\mathbb{P}\Big[\sup_{s\in[t,T]}|U^{t,x,-N}(s)|\ge M\Big]$, which completes the proof of the proposition. $\square$

Now we can finish the proof of part (i) of Theorem 2.6. Fix any $\rho>0$, $N>1$ and $F\in C_b(\mathfrak{X})$. Since $\{T_{M',N}\le T\}$ is a closed subset of $\mathfrak{X}$ for any $N,M>1$, we have

$$\limsup_{M'\to+\infty}\mathfrak{Q}_{M'}[T_{M,N}\le T]\le\mathfrak{Q}[T_{M,N}\le T]. \tag{7.50}$$

Let $M>0$ be chosen as in the statement of Proposition 7.7. We can write then, for any $M'\ge M$

$$\begin{aligned}
\Big|\int_{\mathfrak{X}}Fd\mathfrak{Q}_\varepsilon-\int_{\mathfrak{X}}Fd\mathfrak{Q}\Big| &\le\Big|\int_{T_{M,N}>T}Fd\mathfrak{Q}_\varepsilon-\int_{T_{M,N}>T}Fd\mathfrak{Q}\Big|+\|F\|_\infty(\mathfrak{Q}_\varepsilon[T_{M,N}\le T]+\mathfrak{Q}[T_{M,N}\le T])\\
&\le\Big|\int_{T_{M,N}>T}Fd\mathfrak{Q}_{\varepsilon,M'}-\int_{T_{M,N}>T}Fd\mathfrak{Q}_{M'}\Big|+\|F\|_\infty(\mathfrak{Q}_{\varepsilon,M'}[T_{M,N}\le T]+\rho)\\
&\le\Big|\int_{\mathfrak{X}}Fd\mathfrak{Q}_{\varepsilon,M'}-\int_{\mathfrak{X}}Fd\mathfrak{Q}_{M'}\Big|+\|F\|_\infty(2\mathfrak{Q}_{\varepsilon,M'}[T_{M,N}\le T]+\mathfrak{Q}_{M'}[T_{M,N}\le T]+\rho)
\end{aligned}$$

Passing to the limit, first as $\varepsilon \to 0$, and then $M' \to +\infty$, using (7.48), (7.50), and the convergence $\mathfrak{Q}_{\varepsilon,M'} \Longrightarrow \mathfrak{Q}_{M'}$, we deduce that

$$\limsup_{\varepsilon\to 0}\Big|\int_{\mathfrak{X}} Fd\mathfrak{Q}_\varepsilon - \int_{\mathfrak{X}} Fd\mathfrak{Q}\Big| \le 4\rho\|F\|_\infty.$$

Since $\rho > 0$ can be chosen arbitrarily small, we conclude that

$$\lim_{\varepsilon\to 0}\Big|\int_{\mathfrak{X}} Fd\mathfrak{Q}_\varepsilon - \int_{\mathfrak{X}} Fd\mathfrak{Q}\Big| = 0,$$

which concludes the proof of part (i) of Theorem 2.6.

7.5 Proof of part (ii) of Theorem 2.6

By the tightness of the laws of $(\mathfrak{S}_\varepsilon^{t,x}(\cdot), X_\varepsilon^{t,x}(\cdot))$, for any $\rho > 0$, there exists $N > 1$ such that

$$\mathbb{P}[A_N] < \frac{\rho}{3}, \quad \mathbb{P}[A_{N,\varepsilon}] < \frac{\rho}{3}, \quad \varepsilon \in (0,1],$$

where

$$A_{N,\varepsilon} := \Big[\mathfrak{s}_{T,\varepsilon}^{t,x}(N) \le \|u_0\|_\infty \quad \text{or} \quad \mathfrak{s}_{T,\varepsilon}^{t,x}(-N) \ge -\|u_0\|_\infty\Big],$$

and A_N is defined analogously with $\mathfrak{s}_{T,\varepsilon}^{t,x}$ replaced by $\mathfrak{s}_T^{t,x}$.

According to Proposition 7.7, we can find $M > 1$ such that $\mathbb{P}[B_{N,M}] < \rho/3$, with

$$B_{N,M} := \left[\sup_{(s,u)\in[t,T]\times[-N,N]} |U^{t,x,u}(s)| \ge M, \sup_{s\in[t,T]} |X^{t,x}(s)| \ge M\right].$$

Using part (i) of Theorem 2.6, we conclude that $\mathbb{P}[B_{N,M,\varepsilon}] < \rho/3$, $\varepsilon \in (0,1]$, where $B_{N,M,\varepsilon}$ is defined analogously for $(\mathfrak{S}_\varepsilon^{t,x}(\cdot), X_\varepsilon^{t,x}(\cdot))$. Thanks to equality (7.47) and the fact that

$$\mathfrak{s}_T^{t,x}(u(t,x)) = u_0(X^{t,x}(T)), \quad \mathfrak{s}_{T,\varepsilon}^{t,x}(u_\varepsilon(t,x)) = u_0(X_\varepsilon^{t,x}(T)),$$

we conclude that

$$u_{M'}(t,x) = u(t,x), \quad u_{\varepsilon,M'}(t,x) = u_\varepsilon(t,x), \quad M' \ge M$$

outside $A_N \cup B_{N,M} \cup A_{N,\varepsilon} \cup B_{N,M,\varepsilon}$. Using the already proved convergence in law of $u_{\varepsilon,M'}(t,x)$ to $u_{M'}(t,x)$, as $\varepsilon \to 0$, we conclude from the above that $u_\varepsilon(t,x)$ converges in law to $u(t,x)$, $\varepsilon \to 0$ by the argument presented in Section 7.4.

The convergence of the multi-point statistics follows by essentially the same argument as in the proof of Theorem 2.3. For N distict points $x_1,\ldots,x_N \in \mathbb{R}^d$, $u_1,\ldots,u_N \in \mathbb{R}$, the respective processes

$$\Big((U_\varepsilon^{t,x_1,u_1}(s), X_\varepsilon^{t,x_1}(s)),\ldots,(U_\varepsilon^{t,x_N,u_N}(s), X_\varepsilon^{t,x_N}(s))\Big)_{s\ge t}$$

converge in distribution to

$$\Big((U_1^{t,x_1,u_1}(s), X_1^{t,x_1}(s)),\ldots,(U_N^{t,x_N,u_N}(s), X_N^{t,x_N}(s))\Big)_{s\ge t},$$

where $\Big((U_j^{t,x_j,u_j}(s), X_j^{t,x_j}(s))\Big)_{s\ge t}$, $j = 1,\ldots,N$ are independent copies of solutions of (2.20). This implies that the respective $\mathfrak{s}_T^{t,x_j}(\cdot)$, $j = 1,\ldots,N$ are independent and, as a result, allows us to infer that $\mathscr{U}^{(j)}(t,x_j)$ determined by the corresponding equations (2.21) are also independent. □

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