ANOTHER LOOK INTO THE WONG–ZAKAI THEOREM FOR STOCHASTIC HEAT EQUATION

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Abstract. For the heat equation driven by a smooth, Gaussian random potential:

$$\partial_t u_\varepsilon = \tfrac{1}{2}\Delta u_\varepsilon + u_\varepsilon(\xi_\varepsilon - c_\varepsilon), \quad t > 0, x \in \mathbb{R},$$

where ξ_ε converges to a spacetime white noise, and c_ε is a diverging constant chosen properly, we prove that u_ε converges in L^n to the solution of the stochastic heat equation for any $n \geqslant 1$. Our proof is probabilistic, hence provides another perspective of the general result of Hairer and Pardoux [HP15], for the special case of the stochastic heat equation. We also discuss the transition from homogenization to stochasticity.

1. Introduction and main result

The study of stochastic PDEs has witnessed significant progress in recent years. Several theories have been developed to make sense of singular equations with multiplication of distributions, see [Hai13, Hai14, GIP15, Kup16, OW16] (and the references therein). One example is the Wong–Zakai theorem for stochastic PDEs [HL18, HP15, CS17], which is an infinite dimensional analogue of [WZ65a, WZ65b, SV72]. In this article, we revisit this problem for a special case: the Stochastic Heat Equation (SHE) in one space dimension:

$$\partial_t \mathcal{U} = \tfrac{1}{2}\partial_{xx}\mathcal{U} + \mathcal{U}\xi, \quad t > 0, x \in \mathbb{R}, \tag{1.1}$$

where ξ is a spacetime white noise, built on an underlying probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

The SHE (1.1) has played an important role in the study of directed polymers and random growth phenomena. On the one hand, the solution of (1.1) gives the partition function of a directed polymer in a noisy environment. On the other hand, via the inverse Hopf–Cole transform, the equation (1.1) yields the celebrated Kardar–Parisi–Zhang equation [KPZ86], which describes the height function of a certain type of random growth phenomenon. Even though the physical phenomena described by SHE goes to higher dimensions, we focus on one dimension here. In fact, $d = 2$ and $d > 2$ corresponds to the so-called critical and supercritical cases, and sit beyond any existing theory. There, however, has been works on the cases (in $d \geqslant 2$) where the noise is tuned down to zero suitably with a scaling parameter. See [Fen12, Fen16, CSZ17, GRZ18, MU18].

Throughout this article, we fix a bounded continuous initial condition $u_0(x) \in C_b(\mathbb{R})$, and a mollifier $\phi \in C_c^\infty(\mathbb{R}^2; \mathbb{R}_+)$ with a unit total mass $\int \phi \, dtdx = 1$. Using this mollifier, we construct the mollified noise as

$$\xi_\varepsilon(t,x) = \int_{\mathbb{R}^2} \phi_\varepsilon(t-s, x-y)\xi(s,y)dyds, \quad \phi_\varepsilon(t,x) = \tfrac{1}{\varepsilon^3}\phi(\tfrac{t}{\varepsilon^2}, \tfrac{x}{\varepsilon}). \tag{1.2}$$

Given the smooth function ξ_ε, consider the equation

$$\partial_t u_\varepsilon = \tfrac{1}{2}\partial_{xx} u_\varepsilon + u_\varepsilon(\xi_\varepsilon - c_\varepsilon), \quad t > 0, x \in \mathbb{R}. \tag{1.3}$$

For the analogous equation where ξ_ε is white-in-time, regularized in space, and interpreted in the Itô's sense, Bertini and Cancrini [BC95] showed that, for $c_\varepsilon = 0$, the solution u_ε converges to the solution of the SHE.

When the noise is regularized in both space and time, a non-zero, *divergent* constant $c_\varepsilon \to \infty$ arises. Our main result states that, for a suitable and explicit choice of c_ε, which depends explicitly on ϕ, the solution u_ε of (1.3) converges pointwisely in $L^n(\Omega)$ to the solution of SHE (1.1), for any $n \geqslant 1$. It is a classical result that (1.1) admits a unique (weak and mild) solution starting from $\mathcal{U}(0,x) = u_0(x)$. Also, for fixed $\varepsilon > 0$ and for almost every realization of ξ_ε, it is standard (by Feynman–Kac formula) to show that the PDE (1.3) admits a unique classical solution.

Theorem 1.1. *Let $c_\varepsilon = c_*\varepsilon^{-1} + \frac{1}{2}\sigma_*^2$ with c_*, σ_* given by (2.7) and (2.10). Let u_ε and $\mathcal{U}$ denote the respective solutions of (1.3) and (1.1), both with initial condition $u_0(x)$. Then, for any $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$ and $n \geqslant 1$, the* *random variable $u_\varepsilon(t,x)$ converges in $L^n(\Omega)$ to $\mathcal{U}(t,x)$, i.e.,*

$$\mathbb{E}\big[|u_\varepsilon(t,x)-\mathcal{U}(t,x)|^n\big]\to 0, \quad \text{as } \varepsilon\to 0. \tag{1.4}$$

As mentioned earlier, analogs of Theorem 1.1 have already been established in different settings. Hairer and Pardoux [HP15] established the Wong–Zakai theorem for a general class of semi-linear equations on the torus, with the SHE being a special case. This result was later extended to non-Gaussian noise by Chandra and Shen [CS17], and the problem on the whole line $\mathbb{R}$ was studied by Hairer and Labbé [HL18]. In a related direction, Bailleul, Bernicot, and Frey [BBF17] have studied similar stochastic PDEs via paracontrolled calculus.

All the aforementioned works build on the recently developed theory of regularity structure and paracontrolled calculus [Hai14, GIP15]. In this article, we present a more probabilistic proof of Theorem 1.1. The proof is short, entirely contained within the scope of classical stochastic analysis. This offers a different perspective of the Wong–Zakai theorem for the SHE. For example, the renormalizing constant c_ε is identified in terms of the first and second moments of certain additive functionals of Brownian motions. See the discussion in Section 2.

A related question of interest concerns homogenization of $\partial_t u_\varepsilon = \frac{1}{2}\partial_{xx}u_\varepsilon + \sqrt{\varepsilon}u_\varepsilon\xi_\varepsilon$. For a general class of mixing random potentials, a homogenization result was established by Pardoux and Piatnitski [PP12]. We show in Appendix A how our approach can be adopted to establish homogenization. In fact, we will establish a homogenization result over a range of scales $\alpha\in[1,2)$, together with a Gaussian fluctuation result within this range.

Outline and conventions. In Section 2, we use the Feynman–Kac formula to analyze moments of $u_\varepsilon(t,x)$. These formula are expressed in terms of functionals of some auxiliary Brownian motions. We then establish various properties of these functionals. In Section 3, we prove Theorem 1.1 using the results established in Section 2. This is done by first showing that $u_\varepsilon(t,x)$ is a Cauchy sequence in $L^2(\Omega)$, and then identifying the limit via a Wiener chaos expansion. In Appendix A, we will discuss the homogenization result, prove a central limit theorem, and discuss the transition from the Edwards-Wilkinson to the SHE fluctuations.

Throughout the paper, we denote the Fourier transform of f by

$$\widehat{f}(\xi)=\int_{\mathbb{R}} f(x)e^{-i\xi x}dx.$$

We use $C(a_1,\ldots)$ to denote a generic, deterministic, finite constant that may change from line to line, but depends only on the designated variables $a_1,\ldots$. This is *not* to be confused with the renormalization constant c_ε. Also, we use $r_\varepsilon=(r_\varepsilon(t))_{t\geqslant 0}$ to denote a *generic* (random) process, that uniformly converges to zero, i.e.,

$$\sup_{t\in\mathbb{R}_+}|r_\varepsilon(t)|\leqslant h_\varepsilon\longrightarrow 0, \quad \text{for some deterministic } h_\varepsilon. \tag{1.5}$$

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2. Feynman–Kac Formula and Brownian Functionals

A main tool in this article is the Feynman–Kac formula, which expresses the solution of the PDE (1.3) as

$$u_\varepsilon(t,x)=\mathbb{E}_B\Big[u_0(x+B(t))\exp\Big(\int_0^t\xi_\varepsilon(t-s,x+B(s))ds-c_\varepsilon t\Big)\Big]. \tag{2.1}$$

Here $B(t)$ is a standard Brownian motion starting from the origin, *independent* of the driving noise ξ. In fact, we will be considering several independent Brownian motions. We expand the probability space $(\Omega,\mathscr{F},\mathbb{P})$ to a larger one $(\Omega\times\Sigma,\mathscr{F}\times\mathscr{F}_B,\mathbb{P}\otimes\mathbb{P}_B)$ to include several independent Brownian motions B, B_1, B_2,…, $W,W_1,W_2,\ldots$, independent of ξ. We will use $\mathbb{E}_B$ to denote the expectation on Σ. Also, we will often work with the marginal probability space $(\Omega,\mathscr{F},\mathbb{P})$ or $(\Sigma,\mathscr{F}_B,\mathbb{P}_B)$.

Several functionals of the Brownian motions enter our analysis via (2.1). More precisely, the n-th moment of $u_\varepsilon(t,x)$ is expressed in terms of functionals of Brownian motions. We begin with the first moment. To this end, define the covariance function

$$R(t,x) := \int_{\mathbb{R}^2} \phi(t-s,x-y)\phi(-s,-y)dsdy, \quad R_\varepsilon(t,x) := \tfrac{1}{\varepsilon^3} R(\tfrac{t}{\varepsilon^2},\tfrac{x}{\varepsilon}) = \mathbb{E}[\xi_\varepsilon(t,x)\xi_\varepsilon(0,0)].$$

It is clear that R is an even function. Recall that ϕ is compactly supported. Without loss of generality, throughout this article we assume that ϕ is supported in $(-\frac{1}{2},\frac{1}{2})$ in t, i.e., $\phi(t,\bullet)=0$, $|t|\geqslant \frac{1}{2}$, and hence $R(t,\bullet)=0$, $|t|\geqslant 1$. With $\xi_\varepsilon(t,x)$ being a Gaussian process, averaging over ξ in (2.1) gives

$$\begin{aligned}\mathbb{E}[u_\varepsilon(t,x)] &= \mathbb{E}_B\Big[u_0(x+B(t))\exp\Big(\frac{1}{2}\mathbb{E}\Big[\Big(\int_0^t \xi_\varepsilon(t-s,x+B(s))ds\Big)^2\Big]-c_\varepsilon t\Big)\Big] \\ &= \mathbb{E}_B\Big[u_0(x+B(t))\exp\Big(\int_0^t\int_0^s R_\varepsilon(s-u,B(s)-B(u))duds - c_\varepsilon t\Big)\Big].\end{aligned} \tag{2.2}$$

Before progressing to the formula for higher moments, let us use (2.2) to explain how the renormalizing constant c_ε comes into play. To this end, take $u_0(x)=1$ for simplicity. In this case the solution $\mathcal{U}$ of the limiting SHE (1.1) satisfies $\mathbb{E}[\mathcal{U}(t,x)]=1$. For the convergence in (1.4) to hold, we must choose c_ε so that

$$\mathbb{E}_B\Big[\exp\Big(\int_0^t\int_0^s R_\varepsilon(s-u,B(s)-B(u))duds - c_\varepsilon t\Big)\Big] \longrightarrow 1, \quad \text{as } \varepsilon\to 0.$$

To this end, consider the centered double-integral process

$$X_\varepsilon(t) := \int_0^t\int_0^s R_\varepsilon(s-u,B(s)-B(u))duds - \int_0^t\int_0^s \mathbb{E}_B[R_\varepsilon(s-u,B(s)-B(u))]duds. \tag{2.3}$$

It is more convenient to express X_ε in 'microscopic' coordinates. That is, we use the scaling property $(\varepsilon^{-1}B(\varepsilon^2 t))_{t\geqslant 0} \overset{\text{law}}{=} (B(t))_{t\geqslant 0}$ to write

$$\begin{aligned}X_\varepsilon(t) \overset{\text{law}}{=} X^{\text{mi}}_\varepsilon(t) &:= \varepsilon\int_0^{\varepsilon^{-2}t}\int_0^s R(s-u,B(s)-B(u))duds - \varepsilon\int_0^{\varepsilon^{-2}t}\int_0^s \mathbb{E}_B[R(s-u,B(s)-B(u))]duds \\ &= \varepsilon\int_0^{\varepsilon^{-2}t}\Big(\int_0^s \big(R(u,B(s)-B(s-u)) - \mathbb{E}_B[R(u,B(s)-B(s-u))]\big)du\Big)ds.\end{aligned} \tag{2.4}$$

Since $R(u,x)=0$ whenever $|u|\geqslant 1$, the u-integral in (2.4) goes over $u\in[0,1]$ for all $s\geqslant 1$. Dropping those values of $s<1$ in the integral gives

$$X^{\text{mi}}_\varepsilon(t) = \varepsilon\int_1^{\varepsilon^{-2}t}\mathcal{X}(s)ds + r_\varepsilon(t), \tag{2.5}$$

where, recall that, $r_\varepsilon(t)$ denotes a generic process satisfying (1.5),

$$\mathcal{X}(s) := \int_0^\infty R(u,B(s)-B(s-u))du - c_*, \tag{2.6}$$

$$c_* := \mathbb{E}_B\Big[\int_0^\infty R(u,B(u))du\Big] = \mathbb{E}_B\Big[\int_0^1 R(u,B(s)-B(s-u))du\Big], \tag{2.7}$$

and we take B to be a two-sided Brownian motion in (2.6)–(2.7) so that the resulting expression is defined for all $s\geqslant 0$ (*including* $s\in[0,1]$). It is straightforward to verify that

$$\{\mathcal{X}(s)\}_{s\geqslant 0} \text{ is stationary in } s, \text{ bounded}, \ \mathbb{E}_B[\mathcal{X}(s)]=0, \tag{2.8}$$

$$\text{with } (\mathcal{X}(s))_{s\geqslant s_0}, (\mathcal{X}(s))_{s\leqslant s_0'} \text{ being independent whenever } s_0-s_0'\geqslant 1. \tag{2.9}$$

Thus it is natural to expect $X_\varepsilon(t)$ to converges to $\sigma_* W(t)$, with W being a standard Brownian motion, and

$$\sigma_*^2 := 2\,\mathbb{E}_B\Big[\int_0^\infty \mathcal{X}(s)\mathcal{X}(0)ds\Big]. \tag{2.10}$$

In light of these discussions, we find that $c_\varepsilon := c_*\varepsilon^{-1} + \frac{1}{2}\sigma_*^2$ (as in Theorem 1.1) is the reasonable choice in order for u_ε to converge to the solution $\mathcal{U}$ to the SHE.

Remark 2.1. The constants c_* and σ_*^2, defined in (2.7) and (2.10), can also be expressed in terms of integrals involving the covariance function R and the heat kernel. For example,

$$c_* = \int_0^\infty \mathbb{E}_B\Big[R(u, B(u))\Big]du = \int_0^\infty \int_{\mathbb{R}} R(u,x)\frac{1}{\sqrt{2\pi u}}e^{-\frac{x^2}{2u}}dxdu.$$

Also, the process $X_\varepsilon(t)$ can be expressed as

$$X_\varepsilon(t) = \int_0^t \int_0^s R_\varepsilon(s-u, B(s)-B(u))duds - \frac{c_* t}{\varepsilon} + r_\varepsilon(t).$$

With $X_\varepsilon(t)$ and σ_* defined in the preceding, we rewrite the formula (2.2) in a more compact form as

$$\mathbb{E}[u_\varepsilon(t,x)] = \mathbb{E}_B\Big[u_0(x+B(t))\exp\Big(X_\varepsilon(t) - \tfrac{1}{2}\sigma_*^2 t + r_\varepsilon(t)\Big)\Big]. \tag{2.11}$$

Similar calculations give formulas of higher moments:

$$\mathbb{E}[u_\varepsilon(t,x)^n] = \mathbb{E}_B\Big[\prod_{j=1}^n u_0(x+B_j(t))\exp\Big(\sum_{j=1}^n\Big(X_{j,\varepsilon}(t) - \frac{1}{2}\sigma_*^2 t + r_\varepsilon(t)\Big) + \sum_{1\leqslant i<j\leqslant n} Y_{i,j,\varepsilon}(t)\Big)\Big]. \tag{2.12}$$

Here $B_1, \ldots, B_n$ are independent Brownian motions; the process $X_{j,\varepsilon}(t)$ is obtained by replacing B with B_j in (2.3); and $Y_{i,j,\varepsilon}(t)$ is given by

$$Y_{i,j,\varepsilon}(t) := \int_0^t \int_0^t R_\varepsilon(s-u, B_i(s)-B_j(u))dsdu. \tag{2.13}$$

Indeed, $(X_{j,\varepsilon}(t))_{t\geqslant 0} \overset{\text{law}}{=} (X_\varepsilon(t))_{t\geqslant 0}$. Likewise, writing $Y_\varepsilon(t) := Y_{1,2,\varepsilon}(t)$, we have $(Y_{i,j,\varepsilon}(t))_{t\geqslant 0} \overset{\text{law}}{=} (Y_\varepsilon(t))_{t\geqslant 0}$, for all $i<j$.

We will also need to consider $\mathbb{E}[u_{\varepsilon_1}(t,x)u_{\varepsilon_2}(t,x)]$, i.e., the second moment calculated at different values of ε. A similar calculation gives the formula

$$\mathbb{E}[u_{\varepsilon_1}(t,x)u_{\varepsilon_2}(t,x)] = \mathbb{E}_B\Big[\prod_{j=1}^2 u_0(x+B_j(t))\exp\Big(Y_{\varepsilon_1,\varepsilon_2}(t) + \sum_{j=1}^2\Big(X_{j,\varepsilon_j}(t) - \frac{1}{2}\sigma_*^2 t + r_{\varepsilon_j}(t)\Big)\Big], \tag{2.14}$$

where

$$Y_{\varepsilon_1,\varepsilon_2}(t) := \int_0^t\int_0^t R_{\varepsilon_1,\varepsilon_2}(s-u, B_1(s)-B_2(u))dsdu, \tag{2.15}$$

$$R_{\varepsilon_1,\varepsilon_2}(t,x) := \int_{\mathbb{R}^2}\phi_{\varepsilon_1}(t-s,x-y)\phi_{\varepsilon_2}(-s,-y)dsdy = \mathbb{E}[\xi_{\varepsilon_1}(t,x)\xi_{\varepsilon_2}(0,0)].$$

2.1. **Exponential moments.** We first establish bounds on exponential moments of $X_\varepsilon(t)$ and $Y_{\varepsilon_1,\varepsilon_2}(t)$.

Proposition 2.2. *For any $\lambda, t > 0$, we have*

$$\sup_{\varepsilon\in(0,1)} \mathbb{E}_B[e^{\lambda X_\varepsilon(t)}] + \sup_{\varepsilon_1,\varepsilon_2\in(0,1)} \mathbb{E}_B[e^{\lambda Y_{\varepsilon_1,\varepsilon_2}(t)}] < \infty.$$

Proof. For $X_\varepsilon(t)$, we appeal to the microscopic coordinates, using (2.4)–(2.5) to write

$$X_\varepsilon(t) \overset{\text{law}}{=} X_\varepsilon^{\text{mi}}(t) = \varepsilon\int_1^{\varepsilon^{-2}t}\mathcal{X}(s)ds + r_\varepsilon(t) = \varepsilon\int_1^{[\varepsilon^{-2}t]}\mathcal{X}(s)ds + r_\varepsilon(t).$$

In view of the finite range property (2.9) of $\mathcal{X}$, we decompose

$$X_\varepsilon^{\text{mi}}(t) = \varepsilon\sum_{k\in I_{\text{even}}}\widetilde{\mathcal{X}}_k + \varepsilon\sum_{k\in I_{\text{odd}}}\widetilde{\mathcal{X}}_k,$$

where $\widetilde{\mathcal{X}}_k := \int_k^{k+1}\mathcal{X}(s)ds$, and $I_{\text{even}} := \{1 \leqslant k \leqslant [\varepsilon^2 t]-1, \text{ even}\}$, and $I_{\text{odd}} := \{1\leqslant k\leqslant [\varepsilon^2 t]-1, \text{ odd}\}$. This gives

$$\mathbb{E}_B[e^{\lambda X_\varepsilon(t)}] = \mathbb{E}_B\Big[e^{\lambda r_\varepsilon(t)}\exp\Big(\lambda\varepsilon\sum_{k\in I_{\text{even}}}\widetilde{\mathcal{X}}_k\Big)\exp\Big(\lambda\varepsilon\sum_{k\in I_{\text{odd}}}\widetilde{\mathcal{X}}_k\Big)\Big]$$

$$\leqslant C(\lambda,t)\sqrt{\mathbb{E}_B\Big[\exp\Big(2\lambda\varepsilon\sum_{k\in I_{\text{even}}}\widetilde{\mathcal{X}}_k\Big)\Big]\mathbb{E}_B\Big[\exp\Big(2\lambda\varepsilon\sum_{k\in I_{\text{odd}}}\widetilde{\mathcal{X}}_k\Big)\Big]} = C(\lambda,t)\sqrt{\prod_{k\leqslant[t/\varepsilon^2]-1}\mathbb{E}_B[e^{2\lambda\varepsilon\widetilde{\mathcal{X}}_k}]}.$$

By (2.8), we know that $\widetilde{\mathcal{X}}_k$ is stationary with zero mean and is also uniformly bounded since $R \in C_c^\infty$, so

$$\prod_{k \leqslant [t/\varepsilon^2]-1} \mathbb{E}_B[e^{2\lambda\varepsilon\widetilde{\mathcal{X}}_k}] = \Big(\mathbb{E}_B[e^{2\lambda\varepsilon\widetilde{\mathcal{X}}_1}]\Big)^{[t/\varepsilon^2]-1} \leqslant C(\lambda,t).$$

From this we conclude the desired exponential moment bound on $X_\varepsilon(t)$:

$$\sup_{\varepsilon\in(0,1)} \mathbb{E}_B[e^{\lambda X_\varepsilon(t)}] \leqslant C(\lambda,t) < \infty.$$

We now turn to $Y_{\varepsilon_1,\varepsilon_2}(t)$. Denoting by $\widehat{R}_{\varepsilon_1,\varepsilon_2}$ the Fourier transform of $R_{\varepsilon_1,\varepsilon_2}$in the x-variable, we express $Y_{\varepsilon_1,\varepsilon_2}(t)$ via Fourier transform as

$$Y_{\varepsilon_1,\varepsilon_2}(t) = \int_{[0,t]^2} \left(\int_{\mathbb{R}} (2\pi)^{-1} \widehat{R}_{\varepsilon_1,\varepsilon_2}(s-u,\xi) e^{\mathbf{i}\xi(B_1(s)-B_2(u))} d\xi \right) ds du.$$

Note that $Y_{\varepsilon_1,\varepsilon_2} \geqslant 0$, so from the above expression we calculate the n-th moment of $Y_{\varepsilon_1,\varepsilon_2}(t)$ as

$$\begin{aligned}\mathbb{E}_B[Y_{\varepsilon_1,\varepsilon_2}(t)^n] = \int_{[0,t]^{2n}\times\mathbb{R}^n} &\prod_{j=1}^n \big((2\pi)^{-1}\widehat{R}_{\varepsilon_1,\varepsilon_2}(s_j-u_j,\xi_j)\big) \\ &\times \mathbb{E}_B\Big[\prod_{j=1}^n e^{\mathbf{i}\xi_j B_1(s_j)}\Big]\mathbb{E}_B\Big[\prod_{j=1}^n e^{-\mathbf{i}\xi_j B_2(u_j)}\Big] ds du d\xi.\end{aligned} \tag{2.16}$$

Let us first focus on the integral over $u \in [0,t]^n$. We write

$$\int_{[0,t]^n} \prod_{j=1}^n \widehat{R}_{\varepsilon_1,\varepsilon_2}(s_j-u_j,\xi_j)\mathbb{E}_B\Big[\prod_{j=1}^n e^{-\mathbf{i}\xi_j B_2(u_j)}\Big] du = \int_{[0,t]^n\times\mathbb{R}^n} \prod_{j=1}^n R_{\varepsilon_1,\varepsilon_2}(s_j-u_j,x_j)e^{-\mathbf{i}\xi_j x_j}\mathbb{E}_B\Big[\prod_{j=1}^n e^{-\mathbf{i}\xi_j B_2(u_j)}\Big] du dx.$$

The exponents are purely imaginary. We hence bound those exponentials by 1 in absolute value, and use $0 \leqslant \int_{[0,t]\times\mathbb{R}} R_{\varepsilon_1,\varepsilon_2}(s-u,x) du dx \leqslant 1$ to get

$$\Big| \int_{[0,t]^n} \prod_{j=1}^n \widehat{R}_{\varepsilon_1,\varepsilon_2}(u_j-s_j,\xi_j)\mathbb{E}_B\Big[\prod_{j=1}^n e^{-\mathbf{i}\xi_j B_2(u_j)}\Big] du \Big| \leqslant 1.$$

Inserting this into (2.16) gives

$$\mathbb{E}_B[Y_{\varepsilon_1,\varepsilon_2}(t)^n] \leqslant (2\pi)^{-n} \int_{[0,t]^n\times\mathbb{R}^n} \mathbb{E}_B\Big[\prod_{j=1}^n e^{\mathbf{i}\xi_j B_1(s_j)}\Big] ds d\xi. \tag{2.17}$$

The last integral in (2.17) is in fact the n-th moment of Brownian localtime at the origin. More precisely, let $L(t,x;B)$ denote the localtime process of a Brownian motion B, it is a standard result that

$$(2\pi)^{-n} \int_{[0,t]^n\times\mathbb{R}^n} \mathbb{E}_B\Big[\prod_{j=1}^n e^{\mathbf{i}\xi_j B_1(s_j)}\Big] ds d\xi = \mathbb{E}_B\Big[L(t,0;B_1)^n\Big]. \tag{2.18}$$

Informally speaking, this formula is obtained by interpreting $L(t,0;B_1)$ as $\int_0^t \delta(B_1(s))ds$, where $\delta(\cdot)$ denotes the Dirac function, and taking Fourier transform, similarly to the preceding. The prescribed informal procedure is rigorously implemented by taking a sequence approximating the Dirac function. We omit the details here as the argument is standard.

Now, combine (2.17)–(2.18), and sum over $n \geqslant 0$. We arrive at $\mathbb{E}_B[e^{\lambda Y_{\varepsilon_1,\varepsilon_2}(t)}] \leqslant \mathbb{E}_B[e^{\lambda L(t,0;B_1)}]$. As the Brownian localtime has finite exponential moments, i.e., $\mathbb{E}[e^{\lambda L(t,0;B_1)}] < \infty$ for any $\lambda, t > 0$, we obtain the desired exponential moment bound on $Y_{\varepsilon_1,\varepsilon_2}(t)$:

$$\sup_{\varepsilon_1,\varepsilon_2\in(0,1)} \mathbb{E}_B[e^{\lambda Y_{\varepsilon_1,\varepsilon_2}(t)}] \leqslant C(\lambda,t) < \infty.$$

This completes the proof. □

2.2. **Weak convergence.** In this section, we derive the distributional limit of $X_{j,\varepsilon}$ and $Y_{\varepsilon_1,\varepsilon_2}$. First, since the covariance function $R_{\varepsilon_1,\varepsilon_2}(t,x)$ converges to Dirac function $\delta(t)\delta(x)$, we expect the process $Y_{\varepsilon_1,\varepsilon_2}(t)$ (defined in (2.15)) to converge to the mutual intersection localtime of B_1 and B_2. More precisely, recalling that $L(t,x;B)$ denote the localtime process of a Brownian motion B, we define the mutual intersection localtime of B_1 and B_2 as

$$\ell(t) := L(t,0;B_1-B_2). \tag{2.19}$$

Proposition 2.3. *For any fixed $t>0$, $Y_{\varepsilon_1,\varepsilon_2}(t)\to\ell(t)$ in $L^2(\Sigma)$, as $\varepsilon_1,\varepsilon_2\to 0$.*

Proof. Instead of directly proving the convergence of $Y_{\varepsilon_1,\varepsilon_2}(t)$, let us first consider a modified process $\widetilde{Y}_{\varepsilon_1,\varepsilon_2}(t)$ where B_1 and B_2 are evaluated at the same time:

$$\widetilde{Y}_{\varepsilon_1,\varepsilon_2}(t) := \int_0^t\int_0^t R_{\varepsilon_1,\varepsilon_2}(s-u,B_1(s)-B_2(s))dsdu,$$

and show that $\widetilde{Y}_{\varepsilon_1,\varepsilon_2}(t)$ converges to $\ell(t)$ in $L^2(\Sigma)$. To this end, set

$$F_{\varepsilon_1,\varepsilon_2}(s) := \int_0^t R_{\varepsilon_1,\varepsilon_2}(s-u,B_1(s)-B_2(s))du, \quad s\in[0,t].$$

In the preceding double-integral expression of $\widetilde{Y}_{\varepsilon_1,\varepsilon_2}(t)$, divide the range of integration over s into subintervals depending on its distance from 0 and to t. We rewrite the expression as

$$\widetilde{Y}_{\varepsilon_1,\varepsilon_2}(t) = \int_0^t F_{\varepsilon_1,\varepsilon_2}(s)\Big(\mathbf{1}_{(\varepsilon_1^2+\varepsilon_2^2,t-\varepsilon_1^2-\varepsilon_2^2)}(s)+\mathbf{1}_{[0,\varepsilon_1^2+\varepsilon_2^2]}(s)+\mathbf{1}_{[t-\varepsilon_1^2-\varepsilon_2^2,t]}(s)\Big)ds.$$

Recall that $\phi(t,\bullet)=0$, $|t|\geqslant\frac{1}{2}$. This gives $R_{\varepsilon_1,\varepsilon_2}(s-u,\bullet)=0$ for all $|s-u|\geqslant\varepsilon_1^2+\varepsilon_2^2$. Consequently, for $s\in(\varepsilon_1^2+\varepsilon_2^2,t-\varepsilon_1^2-\varepsilon_2^2)$ we have

$$F_{\varepsilon_1,\varepsilon_2}(s) = \int_{\mathbb{R}} R_{\varepsilon_1,\varepsilon_2}(s-u,B_1(s)-B_2(s))du.$$

Further setting $\Phi(x):=\int_{\mathbb{R}}\phi(t,x)dt$, $\Phi_\varepsilon(x):=\varepsilon^{-1}\Phi(\varepsilon^{-1}x)$, and $\Phi_{\varepsilon_1,\varepsilon_2}(x):=\int\Phi_{\varepsilon_1}(x-y)\Phi_{\varepsilon_2}(-y)dy$, we rewrite the last expression as $F_{\varepsilon_1,\varepsilon_2}(s)=\Phi_{\varepsilon_1,\varepsilon_2}(B_1(s)-B_2(s))$. On the other hand, we also have $|F_{\varepsilon_1,\varepsilon_2}(s)|\leqslant\Phi_{\varepsilon_1,\varepsilon_2}(B_1(s)-B_2(s))$, for all $s\in[0,t]$. This takes into account those values of $s\notin(\varepsilon_1^2+\varepsilon_2^2,t-\varepsilon_1^2-\varepsilon_2^2)$, thereby giving

$$\widetilde{Y}_{\varepsilon_1,\varepsilon_2}(t) = \int_{\varepsilon_1^2+\varepsilon_2^2}^{t-\varepsilon_1^2-\varepsilon_2^2}\Phi_{\varepsilon_1,\varepsilon_2}(B_1(s)-B_2(s))ds + r_{\varepsilon_1,\varepsilon_2}(t), \tag{2.20}$$

where $r_{\varepsilon_1,\varepsilon_2}(t)$ is a remainder term satisfying

$$|r_{\varepsilon_1,\varepsilon_2}(t)| \leqslant \int_0^t \Phi_{\varepsilon_1,\varepsilon_2}(B_1(s)-B_2(s))\Big(\mathbf{1}_{[0,\varepsilon_1^2+\varepsilon_2^2]}(s)+\mathbf{1}_{[t-\varepsilon_1^2-\varepsilon_2^2,t]}(s)\Big)ds.$$

Now, for any interval $[a,b]\subset[0,\infty)$, by the definition of the localtime,

$$\begin{aligned}\int_a^b \Phi_{\varepsilon_1,\varepsilon_2}(B_1(s)-B_2(s))ds &= \int_{\mathbb{R}}\Phi_{\varepsilon_1,\varepsilon_2}(x)\Big(L(b,x;B_1-B_2)-L(a,x;B_1-B_2)\Big)dx\\ &= \int_{\mathbb{R}^2}\Phi(x)\Phi(-y)\Big(L(b,\varepsilon_1x+\varepsilon_2y;B_1-B_2)-L(a,\varepsilon_1x+\varepsilon_2y;B_1-B_2)\Big)dxdy.\end{aligned} \tag{2.21}$$

For almost every realization of B_1-B_2, the function $x\mapsto L(t,x;B_1-B_2)$ is continuous and compactly supported, and the function $t\mapsto L(t,x;B_1-B_2)$ is increasing and continuous. Thus, from (2.20)–(2.21) and the fact that $\int\Phi dx=1$, we conclude that $\widetilde{Y}_{\varepsilon_1,\varepsilon_2}(t)\to\ell(t)=L(t,0;B_1-B_2)$ almost surely as $\varepsilon_1,\varepsilon_2\to 0$. Further, the same calculations (via Fourier transform) as in the proof Proposition 2.2 yields that

$$\sup_{\varepsilon_1,\varepsilon_2}\mathbb{E}_B\big[e^{\lambda\widetilde{Y}_{\varepsilon_1,\varepsilon_2}(t)}\big]\leqslant\mathbb{E}_B\big[e^{\lambda L(t,0;B_1)}\big]<\infty.$$

This property leverages the preceding almost sure convergence into a convergence in $L^2(\Sigma)$:

$$\mathbb{E}_B[(\widetilde{Y}_{\varepsilon_1,\varepsilon_2}(t)-\ell(t))^2]\longrightarrow 0 \quad \text{as } \varepsilon_1,\varepsilon_2\to 0. \tag{2.22}$$

Given (2.22), it remains to show that $Y_{\varepsilon_1,\varepsilon_2}(t) - \widetilde{Y}_{\varepsilon_1,\varepsilon_2}(t) \to 0$ in $L^2(\Sigma)$. We will actually prove the following result: for any choice of $Z_1, Z_2 \in \{Y_{\varepsilon_1,\varepsilon_2}(t), \widetilde{Y}_{\varepsilon_1,\varepsilon_2}(t)\}$, as $\varepsilon_1, \varepsilon_2 \to 0$,

$$\mathbb{E}_B[Z_1 Z_2] \longrightarrow (2\pi)^{-2} \int_{[0,t]^2} \int_{\mathbb{R}^2} \mathbb{E}_B\Big[e^{\mathbf{i}\xi(B_1(s)-B_2(s))} e^{\mathbf{i}\xi'(B_1(s')-B_2(s'))}\Big] d\xi d\xi' ds ds'. \tag{2.23}$$

Once this is done, expanding $\mathbb{E}[(Y_{\varepsilon_1,\varepsilon_2}(t) - \widetilde{Y}_{\varepsilon_1,\varepsilon_2}(t))^2]$ into four terms, and passing to the limit complete the proof.

The proof for all cases of $Z_1, Z_2 \in \{Y_{\varepsilon_1,\varepsilon_2}(t), \widetilde{Y}_{\varepsilon_1,\varepsilon_2}(t)\}$ is the same, and we take $Z_1 = Z_2 = Y_{\varepsilon_1,\varepsilon_2}(t)$ as an example. As in the proof of Proposition 2.2, we express $Y_{\varepsilon_1,\varepsilon_2}(t)$ via Fourier transform as

$$\begin{aligned} Y_{\varepsilon_1,\varepsilon_2}(t) &= \int_{[0,t]^2} \left(\int_{\mathbb{R}} (2\pi)^{-1} \widehat{R}_{\varepsilon_1,\varepsilon_2}(s-u,\xi) e^{\mathbf{i}\xi(B_1(s)-B_2(u))} d\xi \right) ds du \\ &= \int_{[0,t]^2} \left(\int_{\mathbb{R}^2} (2\pi\varepsilon_1^2\varepsilon_2^2)^{-1} \widehat{\phi}(\tfrac{s-u-w}{\varepsilon_1^2}, \varepsilon_1\xi) \widehat{\phi}(\tfrac{-w}{\varepsilon_2^2}, -\varepsilon_2\xi) e^{\mathbf{i}\xi(B_1(s)-B_2(u))} d\xi dw \right) ds du, \end{aligned}$$

where $\widehat{\phi}$ denotes the Fourier transform of ϕ in the x-variable. Squaring the last expression and taking expectation gives

$$\mathbb{E}_B[Y_{\varepsilon_1,\varepsilon_2}(t)^2] = \int_{[0,t]^2} G_{\varepsilon_1,\varepsilon_2}(s,s') ds ds',$$

where

$$\begin{aligned} G_{\varepsilon_1,\varepsilon_2}(s,s') := \int_{[0,t]^2} \int_{\mathbb{R}^4} &(2\pi\varepsilon_1^2\varepsilon_2^2)^{-2} \widehat{\phi}(\tfrac{s-u-w}{\varepsilon_1^2}, \varepsilon_1\xi) \widehat{\phi}(\tfrac{-w}{\varepsilon_2^2}, -\varepsilon_2\xi) \widehat{\phi}(\tfrac{s'-u'-w'}{\varepsilon_1^2}, \varepsilon_1\xi') \widehat{\phi}(\tfrac{-w'}{\varepsilon_2^2}, -\varepsilon_2\xi') \\ &\times \mathbb{E}_B\Big[e^{\mathbf{i}\xi(B_1(s)-B_2(u))} e^{\mathbf{i}\xi'(B_1(s')-B_2(u'))}\Big] d\xi d\xi' dw dw' du du'. \end{aligned}$$

Fix $(s,s') \in (0,t)^2$. Recall that $\widehat{\phi}(t,\cdot) = 0$ whenever $|t| \geqslant \frac{1}{2}$, and note that, with $(s,s') \in (0,t)^2$ being fixed, the conditions $(t-s), (t-s'), s, s' \geqslant \frac{1}{2}(\varepsilon_1^2 + \varepsilon_2^2)$ holds for all small enough $\varepsilon_1, \varepsilon_2$. Consequently, in the last expression of $G_{\varepsilon_1,\varepsilon_2}(s,s')$, for all $\varepsilon_1, \varepsilon_2$ small enough, the integration domain of u, u' can be (and is) replaced $[0,t]^2 \mapsto \mathbb{R}^2$. A change of variables in this case yields

$$\begin{aligned} G_{\varepsilon_1,\varepsilon_2}(s,s') = \int_{\mathbb{R}^6} &(2\pi)^{-2} \widehat{\phi}(u, \varepsilon_1\xi) \widehat{\phi}(-w, -\varepsilon_2\xi) \widehat{\phi}(u', \varepsilon_1\xi') \widehat{\phi}(-w', -\varepsilon_2\xi') \\ &\times \mathbb{E}_B\Big[e^{\mathbf{i}\xi(B_1(s)-B_2(s-\varepsilon_1^2 u - \varepsilon_2^2 w))} e^{\mathbf{i}\xi'(B_1(s')-B_2(s'-\varepsilon_1^2 u' - \varepsilon_2^2 w'))}\Big] d\xi d\xi' dw dw' du du'. \end{aligned}$$

Observe that the expectation in the above expression has Gaussian tails in ξ, ξ', so by the dominated convergence theorem and the fact that $\int \widehat{\phi}(u,0) du = \int \phi du dx = 1$, we obtain

$$G_{\varepsilon_1,\varepsilon_2}(s,s') \longrightarrow (2\pi)^{-2} \int_{\mathbb{R}^2} \mathbb{E}_B\Big[e^{\mathbf{i}\xi(B_1(s)-B_2(s))} e^{\mathbf{i}\xi'(B_1(s')-B_2(s'))}\Big] d\xi d\xi', \quad \text{pointwisely in } (0,t)^2. \tag{2.24}$$

To achieve (2.23), we need to upgrade the pointwise convergence of (2.24) to convergence in $L^1([0,t]^2)$. To this end, with $|\widehat{\phi}(\cdot,\xi)| \leqslant \widehat{\phi}(\cdot,0)$, we bound

$$\begin{aligned} |G_{\varepsilon_1,\varepsilon_2}(s,s')| \leqslant \int_{\mathbb{R}^6} &(2\pi\varepsilon_1^2\varepsilon_2^2)^{-2} \widehat{\phi}(\tfrac{s-u-w}{\varepsilon_1^2}, 0) \widehat{\phi}(\tfrac{-w}{\varepsilon_2^2}, 0) \widehat{\phi}(\tfrac{s'-u'-w'}{\varepsilon_1^2}, 0) \widehat{\phi}(\tfrac{-w'}{\varepsilon_2^2}, 0) \\ &\times \mathbb{E}_B\Big[e^{\mathbf{i}\xi B_1(s)} e^{\mathbf{i}\xi' B_1(s')}\Big] d\xi d\xi' dw dw' du du'. \end{aligned}$$

After integrating in w, w', u, u' on the RHS of the last integral, we have

$$|G_{\varepsilon_1,\varepsilon_2}(s,s')| \leqslant (2\pi)^{-2} \int_{\mathbb{R}^2} \mathbb{E}_B\Big[e^{\mathbf{i}\xi B_1(s)} e^{\mathbf{i}\xi' B_1(s')}\Big] d\xi d\xi' \leqslant \frac{C}{\sqrt{(s \wedge s')|s-s'|}} \in L^1([0,t]^2).$$

Given this, the dominated convergence theorem upgrades (2.24) into a convergence in $L^1([0,t]^2)$. This gives (2.23) and hence completes the proof. $\square$

We next turn to the distributional limit of $X_{1,\varepsilon}$ and $X_{2,\varepsilon}$ (defined in (2.3)). Hereafter, we use $\Rightarrow$ to denote the weak convergence of probability laws in a designated space, and endow the space $C[0,\infty)$ with the topology of uniform convergence over compact subsets of $[0,\infty)$.

Proposition 2.4. *As $\varepsilon \to 0$,*

$$(B_1, B_2, \ell, X_{1,\varepsilon}, X_{2,\varepsilon}) \Longrightarrow (B_1, B_2, \ell, \sigma_* W_1, \sigma_* W_2) \quad \text{in } (C[0,\infty))^5. \tag{2.25}$$

where W_1, W_2 are standard Brownian motions independent of (B_1, B_2), and $\sigma_* \in (0,\infty)$ is given in (2.10).

Proof. The proof consists of two steps.
Step 1: Instead of showing (2.25) directly, let us first establish

$$(B_1, B_2, X_{1,\varepsilon}, X_{2,\varepsilon}) \Longrightarrow (B_1, B_2, \sigma_* W_1, \sigma_* W_2) \quad \text{in } (C[0,\infty))^4. \tag{2.26}$$

To this end, we appeal to microscopic coordinates. That is, with $X^{\mathrm{mi}}_{\varepsilon,j}$ given in (2.4) (with B replaced by B_j), we have

$$(B_1(t), B_2(t), X_{1,\varepsilon}(t), X_{2,\varepsilon}(t))_{t\geqslant 0} \overset{\text{law}}{=} (\varepsilon B_1(\varepsilon^{-2}t), \varepsilon B_2(\varepsilon^{-2}t), X^{\mathrm{mi}}_{1,\varepsilon}(t), X^{\mathrm{mi}}_{2,\varepsilon}(t))_{t\geqslant 0}. \tag{2.27}$$

We begin by writing $X^{\mathrm{mi}}_{j,\varepsilon}$ in terms of stochastic integrals. Let $D_{j,r}$ denote the Malliavin derivative with respect to $dB_j(r)$ on $(\Sigma, \mathscr{F}_B, \mathbb{P}_B)$, and let $\mathscr{F}_j(r)$ denote the canonical filtration of B_j. The Clark–Ocone formula [Nua06, Proposition 1.3.14] states that (with $X^{\mathrm{mi}}_{j,\varepsilon}(t)$ having zero mean)

$$X^{\mathrm{mi}}_{j,\varepsilon}(t) = \varepsilon \int_0^{\varepsilon^{-2}t} \mathcal{Z}_{j,\varepsilon}(r,t) dB_j(r), \quad \mathcal{Z}_{j,\varepsilon}(r,t)(r,t) := \varepsilon^{-1}\mathbb{E}[D_{j,r} X^{\mathrm{mi}}_{j,\varepsilon}(t) | \mathscr{F}_j(r)].$$

A direct calculation yields

$$D_{j,r} X^{\mathrm{mi}}_{j,\varepsilon}(t) = \varepsilon \int_0^{\varepsilon^{-2}t} \int_0^s \partial_x R(s-u, B_j(s) - B_j(u)) duds \mathbf{1}_{\{u\leqslant r<s\}},$$

so

$$\mathcal{Z}_{j,\varepsilon}(r,t) = \int_r^{\varepsilon^{-2}t} \int_0^r \mathbb{E}_B\Big[\partial_x R(s-u, B_j(s) - B_j(u))\Big| \mathscr{F}_j(r)\Big] duds. \tag{2.28}$$

The function $\partial_x R(s,x)$ vanishes for all $|s| \geqslant 1$ (because R does), so by defining

$$\widetilde{\mathcal{Z}}_j(r) := \mathbf{1}\{r \geqslant 1\} \int_r^{r+1} \int_{r-1}^r \mathbb{E}_B\Big[\partial_x R(s-u, B_j(s) - B_j(u))\Big| \mathscr{F}_j(r)\Big] duds, \tag{2.29}$$

we have $\mathcal{Z}_{j,\varepsilon}(r,t) = \widetilde{\mathcal{Z}}_j(r)$, for all $r \in [1, \varepsilon^{-2}t - 1]$. The latter is preferred for our purpose, because it does not depend on t. In particular, the analogous integrated process:

$$t \longmapsto \widetilde{X}^{\mathrm{mi}}_{j,\varepsilon}(t) := \varepsilon \int_0^{\varepsilon^{-2}t} \widetilde{\mathcal{Z}}_j(r) dB_j(r)$$

is a martingale (unlike $X^{\mathrm{mi}}_{j,\varepsilon}(t)$, which is not due to the t-dependence of $\mathcal{Z}_{j,\varepsilon}(r,t)$). Also, for each $t \in \mathbb{R}_+$, the $L^2(\Sigma)$-distance between $\widetilde{X}^{\mathrm{mi}}_{j,\varepsilon}$ and $X^{\mathrm{mi}}_{j,\varepsilon}$ vanishes as $\varepsilon \to 0$:

$$\mathbb{E}_B[|\widetilde{X}^{\mathrm{mi}}_{j,\varepsilon}(t) - X^{\mathrm{mi}}_{j,\varepsilon}(t)|^2] = \varepsilon^2 \int_0^{\varepsilon^{-2}t} \mathbb{E}_B[|\mathcal{Z}_{j,\varepsilon}(r,t) - \widetilde{\mathcal{Z}}_j(r)|^2] dr \leqslant C\varepsilon^2. \tag{2.30}$$

Given this, let us focus on the modified process $\widetilde{X}^{\mathrm{mi}}_{j,\varepsilon}$ instead of $X^{\mathrm{mi}}_{j,\varepsilon}$.

Now, consider the $C([0,\infty), \mathbb{R}^4)$-valued process

$$M_\varepsilon(t) = (M_{j,\varepsilon}(t))_{j=1}^4 := (\varepsilon B_1(\varepsilon^{-2}t), \varepsilon B_2(\varepsilon^{-2}t), \widetilde{X}^{\mathrm{mi}}_{1,\varepsilon}(t), \widetilde{X}^{\mathrm{mi}}_{2,\varepsilon}(t)).$$

It is a continuous martingale, with cross-variance

$$\langle M_{i,\varepsilon}, M_{j,\varepsilon}\rangle(t) = \begin{cases} 0 & \text{, for } (i,j) = (1,2), (3,4), (1,4), (2,3), \\ \varepsilon^2 \int_0^{\varepsilon^{-2}t} \widetilde{\mathcal{Z}}_i(s) ds & \text{, for } (i,j) = (1,3), (2,4), \\ t & \text{, for } (i,j) = (1,1), (2,2), \\ \varepsilon^2 \int_0^{\varepsilon^{-2}t} \widetilde{\mathcal{Z}}_i^2(s) ds, & \text{for } (i,j) = (3,3), (4,4). \end{cases}$$

Further, straightforward calculations from the expression (2.29) gives, with $\partial_x R$ being an odd function in x, $\mathbb{E}_B[\widetilde{\mathcal{Z}}_i(s)] = 0$ and $\mathbb{E}_B[\widetilde{\mathcal{Z}}_i^2(s)] = (\sigma'_*)^2$, for all $s \geqslant 1$, where

$$(\sigma'_*)^2 := \mathbb{E}_B[\widetilde{\mathcal{Z}}_i^2(1)] = \mathbb{E}_B\bigg[\mathbb{E}_B\Big[\int_1^2 \int_0^1 \partial_x R(s-u, B_j(s) - B_j(u))duds\Big|\mathscr{F}_j(1)\Big]^2\bigg]. \tag{2.31}$$

Calculating the conditional expectation in (2.29) gives

$$\widetilde{\mathcal{Z}}_j(r) = \mathbf{1}\{r \geqslant 1\} \int_r^{r+1} \int_{r-1}^{r} \widetilde{R}(s, r, u, B_j(r) - B_j(u))duds,$$

where $\widetilde{R}(s, r, u, x) := \int_{\mathbb{R}} q(s-r, x-y)\partial_y R(s-u, y)dy$, and $q(t, x) := \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}$ denotes the standard heat kernel. From this expression, it is readily check that $(\widetilde{\mathcal{Z}}_i(s))_{s\geqslant 1}$ is bounded, stationary, and has a finite range of dependence similar to (2.9). In particular, the process $\{B_j(r) - B_j(u) : u \in [r-1, r]\}_{r\geqslant 0}$ is ergodic. Consequently, Birkhoff's Ergodic theorem applied to $(\widetilde{\mathcal{Z}}_i(s))_{s\geqslant 1}$ and $\{\widetilde{\mathcal{Z}}_i^2(s)\}_{s\geqslant 1}$ gives

$$\varepsilon^2 \int_0^{\varepsilon^{-2}t} \widetilde{\mathcal{Z}}_i(s)ds \longrightarrow 0, \quad \varepsilon^2 \int_0^{\varepsilon^{-2}t} \widetilde{\mathcal{Z}}_i^2(s)ds \longrightarrow (\sigma'_*)^2 t,$$

almost surely as $\varepsilon \to 0$, for any fixed $t \in \mathbb{R}_+$. Given these properties, the martingale central limit theorem [EK86, Thm 1.4, p339] yields that $M \Rightarrow (B_1, B_2, \sigma'_* W_1, \sigma'_* W_2)$, in $(C[0,\infty))^4$. This together with (2.27) and (2.30) gives

$$(B_1(t), B_2(t), X_{1,\varepsilon}(t), X_{2,\varepsilon}(t)) \Longrightarrow (B_1(t), B_2(t), \sigma'_* W_1(t), \sigma'_* W_2(t)), \text{ in fdd.} \tag{2.32}$$

Given (2.32), it now suffices to establish the tightness of $(B_1, B_2, X_{1,\varepsilon}, X_{2,\varepsilon})$ in $(C[0,\infty))^4$, and show that $(\sigma'_*)^2 = \sigma_*^2$. The first step is to appeal to microscopic coordinates. Using (2.4)–(2.5) we write

$$X_{j,\varepsilon}(t) \stackrel{\text{law}}{=} X^{\text{mi}}_{j,\varepsilon}(t) = \varepsilon \int_1^{\varepsilon^{-2}t} \mathcal{X}_j(s)ds + r_\varepsilon(t).$$

Given the properties (2.8)–(2.9) of $\mathcal{X}_j$, a classical functional central limit theorem, see, e.g., [Bil99, pp 178–179], asserts that

$$X_{j,\varepsilon} \Longrightarrow \sigma_* W_j, \quad \text{in } C[0,\infty). \tag{2.33}$$

To apply the result in [Bil99], a $\varphi-$mixing condition needs to be checked [Bil99, Eqn (20.65)]. This is clearly satisfied in our case because $\mathcal{X}_j$ has a finite range of dependence. The convergence in (2.33) in particular implies the tightness of $(B_1, B_2, X_{1,\varepsilon}, X_{2,\varepsilon})$. Further, comparing (2.32)–(2.33), we see that $\sigma_*^2 = (\sigma'_*)^2$ must holds. (Alternatively, it is possibly to show $\sigma_*^2 = (\sigma'_*)^2$ by calculations from the expressions (2.10) and (2.31).) We thus conclude (2.26).

Step 2: Having established (2.26), our next goal is to extend the convergence result to include the localtime process ℓ. First, Tanaka's formula gives

$$2\ell(t) = |B_1(t) - B_2(t)| - \int_0^t \operatorname{sgn}(B_1(s) - B_2(s))d(B_1 - B_2)(s).$$

Had it been the case that the RHS were a continuous function of $B_1 - B_2$, the desired result (2.25) would follow immediately from (2.26). We show in Lemma 2.5 that, in fact, the stochastic integral $\int_0^t \operatorname{sgn}(B_1(s) - B_2(s))d(B_1 - B_2)(s)$ is well-approximated by a sequence of continuous functions of $B_1 - B_2$. That is, there exists a sequence $\{f_n\}_{n\geqslant 1} \subset C(C[0,\infty); C[0,\infty))$ such that

$$\Big\| \int_0^\bullet \operatorname{sgn}(B_1(s) - B_2(s))d(B_1 - B_2)(s) - f_n(B_1 - B_2)\Big\|_{C[0,\infty)} \longrightarrow 0 \text{ in probability, as } n \to \infty. \tag{2.34}$$

Now, fix arbitrary bounded and continuous $g : (C[0,\infty))^3 \to \mathbb{R}$ and $h : (C[0,\infty))^2 \to \mathbb{R}$, and consider test functions of the type $g \otimes h \in (C[0,\infty))^5 \to \mathbb{R}$. It is known that the linear span of functions of this type is dense in $C((C[0,\infty))^5; \mathbb{R})$. Hence proving (2.25) amounts to proving

$$\mathbb{E}_B[g(B_1, B_2, \ell)h(X_{1,\varepsilon}, X_{2,\varepsilon})] \longrightarrow \mathbb{E}_B[g(B_1, B_2, \ell)h(\sigma_* W_1, \sigma_* W_2)], \quad \text{as } \varepsilon \to 0. \tag{2.35}$$

Set $2\ell_n(t) := |B_1(t) - B_2(t)| - f_n(B)(t)$. Since f_n is continuous, for each fixed n, from (2.26) we have

$$\mathbb{E}_B[g(B_1, B_2, \ell_n)h(X_{1,\varepsilon}, X_{2,\varepsilon})] \longrightarrow \mathbb{E}_B[g(B_1, B_2, \ell_n)h(\sigma_* W_1, \sigma_* W_2)] \quad \text{as } \varepsilon \to 0.$$

On the other hand, with g being bounded and continuous, by (2.34), we have

$$\mathbb{E}_B[|g(B_1, B_2, \ell_n) - g(B_1, B_2, \ell)|] \to 0,$$

as $n \to \infty$. From these the desired result (2.35) follows. The proof is complete. $\square$

Lemma 2.5. *The claim* (2.34) *holds for a sequence* $\{f_n\}_{n\geqslant 1} \subset C(C[0,\infty); C[0,\infty))$.

Proof. Set $\overline{B}(t) := B_1(t) - B_2(t)$ and $U(t) := \int_0^t \mathrm{sgn}(\overline{B}(s))d\overline{B}(s)$ to simplify the notation. We begin by constructing the continuous function f_n. Set $\zeta(x) := x\mathbf{1}\{|x| \leqslant 1\} + \mathrm{sgn}(x)\mathbf{1}\{|x| > 1\}$, $\zeta_n(x) := \zeta(n^{1/4}x)$, and define

$$f_n(y)(t) := \int_0^t \sum_{k=0}^{\infty} \zeta_n(y(\tfrac{k}{n}))\mathbf{1}_{[\frac{k}{n},\frac{k+1}{n})}(s)dy(s) := \sum_{k=0}^{\infty} \zeta_n(y(\tfrac{k}{n}))\big(y(\tfrac{k+1}{n} \wedge t) - y(\tfrac{k}{n} \wedge t)\big).$$

Indeed, f_n is continuous for each fixed n. Fix an arbitrary $T > 0$. Using Doob's L^2-martingale inequality and Itô isometry, we calculate

$$\begin{aligned}\mathbb{E}_B\Big[\sup_{[0,T]} |U(t) - f_n(\overline{B})(t)|^2\Big] &\leqslant C\int_0^T \mathbb{E}_B\Big[\Big|\sum_{k=0}^{\infty} \zeta_n(\overline{B}(\tfrac{k}{n}))\mathbf{1}_{[\frac{k}{n},\frac{k+1}{n})}(t) - \mathrm{sgn}(\overline{B}(t))\Big|^2\Big]dt \\ &= C\sum_{k=0}^{\infty}\int_{I_{k,n}(T)} \mathbb{E}_B[|\zeta_n(\overline{B}(\tfrac{k}{n})) - \mathrm{sgn}(\overline{B}(t))|^2]dt,\end{aligned} \tag{2.36}$$

where $I_{k,n}(T) := [\frac{k}{n}, \frac{k+1}{n}) \cap [0,T]$. Set $V_{k,n} := \sup_{s\in[0,\frac{1}{n}]} |\overline{B}(s + \frac{k}{n}) - \overline{B}(\frac{k}{n})|$. On the interval $t \in I_{k,n}(T)$, we have $\zeta_n(\overline{B}(\frac{k}{n})) = \mathrm{sgn}(\overline{B}(t))$ whenever $|\overline{B}(\frac{k}{n})| > n^{-\frac{1}{4}}$ and $V_{k,n} < n^{-\frac{1}{4}}$. Hence

$$\mathbb{E}_B\Big[\sup_{[0,T]} |U(t) - f_n(\overline{B})(t)|^2\Big] \leqslant C\sum_{k=0}^{[nT]}\Big(\mathbb{P}_B\big[V_{k,n} \geqslant n^{-\frac{1}{4}}\big] + \mathbb{P}_B\big[|\overline{B}(\tfrac{k}{n})| \leqslant n^{-\frac{1}{4}}\big]\Big)\frac{1}{n}.$$

By the scaling property of Brownian motion and the reflection principle, we have that $V_{k,n} \overset{\text{law}}{=} \sqrt{2/n}|Z|$ and that $\overline{B}(\frac{k}{n}) \overset{\text{law}}{=} \sqrt{2k/n}Z$, where Z is a standard Gaussian. This gives

$$\mathbb{E}_B\Big[\sup_{[0,T]} |U(t) - f_n(\overline{B})(t)|^2\Big] \leqslant C\sum_{k=0}^{[nT]}\Big(\mathbb{P}_B\big[|Z| \geqslant 2^{-\frac{1}{2}}n^{\frac{1}{4}}\big] + \mathbb{P}_B\big[|Z| \leqslant \tfrac{n^{1/4}}{(2k)^{1/2}}\big]\Big)\frac{1}{n},$$

with $\frac{n^{1/4}}{(2k)^{1/2}} := \infty$ when $k = 0$. It is now readily verified that the last expression tends to 0 as $n \to \infty$. From this the desired result follows: $\sup_{[0,T]} |U(t) - f_n(\overline{B})(t)| \to 0$ in probability, as $n \to \infty$, for each fixed T. $\square$

3. Proof of Theorem 1.1

Let us first establish the boundedness of moments of $u_\varepsilon(t,x)$. Set $\lambda_n := \frac{n(n+1)}{2}$. With the initial condition u_0 being bounded, applying Hölder's inequality (with exponents $(\lambda_n, \ldots, \lambda_n)$) in the formula (2.12), we have

$$\begin{aligned}\mathbb{E}[|u_\varepsilon(t,x)|^n] &\leqslant C\bigg(\prod_{j=1}^n \mathbb{E}_B\big[\exp\big(\lambda_n\big(X_{j,\varepsilon}(t) - \tfrac{1}{2}\sigma_*^2 t + r_\varepsilon(t)\big)\big)\big]\prod_{1\leqslant i<j\leqslant n}\mathbb{E}_B\big[\exp\big(\lambda_n Y_{i,j,\varepsilon}(t)\big)\big]\bigg)^{1/\lambda_n} \\ &\leqslant C\bigg(\mathbb{E}_B\big[\exp\big(\lambda_n X_\varepsilon(t)\big)\big]\bigg)^{n/\lambda_n}\bigg(\mathbb{E}_B\big[\exp\big(\lambda_n Y_{i,j,\varepsilon}(t)\big)\big]\bigg)^{n(n-1)/2\lambda_n}.\end{aligned}$$

Using the exponential moment bounds from Proposition 2.2, we obtain $\sup_{\varepsilon\in(0,1)} \mathbb{E}[|u_\varepsilon(t,x)|^n] < \infty$. That is, moments of $u_\varepsilon(t,x)$ are bounded uniformly in ε. This reduces proving (1.4) for all $n \geqslant 1$ to proving (1.4) for *just one* $n \geqslant 1$, since it implies the convergence in probability, and combining with the uniform integrability of $|u_\varepsilon(t,x)|^n$, we will have the convergence in $L^n(\Omega)$. We henceforward consider $n = 2$.

Let us first identify the limit of $\mathbb{E}[u_{\varepsilon_1}(t,x)u_{\varepsilon_2}(t,x)]$. Recall from (2.19) that $\ell(t)$ denotes the mutual intersection localtime of B_1, B_2, and that $\mathcal{U}$ denotes the solution of the SHE (1.1).

Proposition 3.1. *We have*

$$\lim_{\varepsilon_1,\varepsilon_2\to 0}\mathbb{E}[u_{\varepsilon_1}(t,x)u_{\varepsilon_2}(t,x)] = \mathbb{E}_B\Big[u_0(x+B_1(t))u_0(x+B_2(t))\exp(\ell(t))\Big].$$

Proof. The starting point of the proof is the formula (2.14):

$$\mathbb{E}[u_{\varepsilon_1}(t,x)u_{\varepsilon_2}(t,x)] = \mathbb{E}_B\Big[\prod_{j=1}^{2}u_0(x+B_j(t))\exp\Big(Y_{\varepsilon_1,\varepsilon_2}(t)+\sum_{j=1}^{2}\Big(X_{j,\varepsilon_j}(t)-\frac{1}{2}\sigma_*^2 t + r_{\varepsilon_j}(t)\Big)\Big)\Big]. \tag{3.1}$$

By virtue of Propositions 2.3–2.4, we have

$$(B_1(t),B_2(t),Y_{\varepsilon_1,\varepsilon_2}(t),X_{1,\varepsilon_1}(t),X_{2,\varepsilon_2}(t)) \Rightarrow (B_1(t),B_2(t),\ell(t),\sigma_*W_1(t),\sigma_*W_2(t))$$

in distribution. The proof is complete by invoking Propositions 2.2. ☐

Now, with

$$\mathbb{E}[(u_{\varepsilon_1}(t,x)-u_{\varepsilon_2}(t,x))^2] = \mathbb{E}[u_{\varepsilon_1}(t,x)u_{\varepsilon_1}(t,x)] - 2\mathbb{E}[u_{\varepsilon_1}(t,x)u_{\varepsilon_2}(t,x)] + \mathbb{E}[u_{\varepsilon_2}(t,x)u_{\varepsilon_2}(t,x)],$$

Proposition 3.1 has an immediate corollary:

Corollary 3.2. *The sequence $\{u_\varepsilon(t,x)\}_{\varepsilon\in(0,1)}$ is Cauchy in $L^2(\Omega)$.*

Given this result, it suffices to identify the unique limit of $u_\varepsilon(t,x)$ in $L^2(\Omega)$. We achieve this by *Wiener chaos expansion.* Fix $(t,x)\in\mathbb{R}_+\times\mathbb{R}$ hereafter, and denote $R^k_< := \{(s_1,\ldots,s_k)\in\mathbb{R}^k : s_1<\ldots<s_k\}$. Given any $f\in L^2(R^k_<\times\mathbb{R}^k)$, we consider the k-th order multiple stochastic integral

$$I_k(f) := \int_{\mathbb{R}^k_<\times\mathbb{R}^k} f(s_1,\ldots,s_k,y_1,\ldots,y_k)\prod_{i=1}^{n}\xi(s_i,y_i)ds_i dy_i.$$

Let $q(s,y)=\frac{1}{\sqrt{2\pi s}}e^{-y^2/2s}$ denotes the standard heat kernel, the solution $\mathcal{U}$ of the SHE permits the chaos expansion (we omit the dependence on (t,x))

$$\mathcal{U}(t,x)=\sum_{k=0}^{\infty}I_k(f_k),\quad f_k := \mathbf{1}_{\{0<s_1<\ldots<s_k<t\}}\int_{\mathbb{R}}u_0(y_0)\prod_{i=0}^{k}q(s_{i+1}-s_i,y_{i+1}-y_i)dy_0, \tag{3.2}$$

under the convention $y_{k+1}:=x$, $s_0:=0$, and $s_{k+1}:=t$. For u_ε, a similar expansion also exists: using the Stroock formula [Str87, Eqn (7), p3], we arrive at

$$u_\varepsilon(t,x)=\sum_{k=0}^{\infty}I_k(f_{k,\varepsilon}),\quad f_{k,\varepsilon} := \mathbb{E}[D^k u_\varepsilon(t,x)]. \tag{3.3}$$

(Alternatively, this formula (3.3) can also be obtained from the Wick exponential.) Here D denotes the Malliavin derivative with respect to ξ on $(\Omega,\mathscr{F},\mathbb{P})$. To calculate the chaos coefficient $f_{k,\varepsilon}$, we set

$$\Psi_{\varepsilon,B}(r,y) := \int_0^t \phi_\varepsilon(t-s-r,x+B(s)-y)ds, \tag{3.4}$$

and rewrite the Feynman–Kac formula (2.1) as

$$u_\varepsilon(t,x)=\mathbb{E}_B\Big[u_0(x+B(t))\exp\Big(\int_{\mathbb{R}^2}\Psi_{\varepsilon,B}(r,y)\xi(r,y)dydr - c_\varepsilon t\Big)\Big].$$

From this expression we calculate

$$\begin{aligned} &f_{k,\varepsilon}(r_1,\ldots,r_k,y_1,\ldots,y_k) \\ &= \mathbb{E}_B\Big[u_0(x+B(t))\exp\Big(\int_0^t\int_0^s R_\varepsilon(s-u,B(s)-B(u))duds - c_\varepsilon t\Big)\prod_{i=1}^{k}\Psi_{\varepsilon,B}(r_i,y_i)\Big]. \end{aligned} \tag{3.5}$$

Denoting the $L^2(\Omega)-$limit of $u_\varepsilon(t,x)$ by $\mathscr{U}(t,x)$, and the chaos expansion of $\mathscr{U}(t,x)$ is written as

$$\mathscr{U}(t,x)=\sum_{k=0}^{\infty}I_k(\widetilde{f}_k). \tag{3.6}$$

The following lemma completes the proof of Theorem 1.1.

Lemma 3.3. *$\mathscr{U}(t,x) = \mathcal{U}(t,x)$ in $L^2(\Omega)$.*

Remark 3.4. The major component of the following proof is to establish the convergence $f_{k,\varepsilon} \to f_k$ in $L^2(\mathbb{R}^k_< \times \mathbb{R}^k)$. To set up the premise of the proof, we first give a heuristic explanation why the convergence should hold. Under current notations, Proposition 2.4 gives the following weak convergence in $C[0,\infty)^2$,

$$\Big(\int_0^t \int_0^s R_\varepsilon(s-u, B(s)-B(u))duds - c_\varepsilon t, B(t)\Big) \Longrightarrow \Big(\sigma_* W(t) - \tfrac{1}{2}\sigma_*^2 t, B(t)\Big), \tag{3.7}$$

where W and B are *independent* standard Brownian motions. This implies

$$\mathbb{E}_B\Big[u_0(x+B(t)) \exp\Big(\int_0^t \int_0^s R_\varepsilon(s-u, B(s)-B(u))duds - c_\varepsilon t\Big) F(B)\Big] \longrightarrow \mathbb{E}_B\big[u_0(x+B(t))F(B)\big], \tag{3.8}$$

for any continuous, bounded test function $F : C[0,\infty) \to \mathbb{R}$. Referring to (3.5) and (3.2), together with $\phi_\varepsilon(t,x) \to \delta(t,x)$, we can informally view the convergence $f_{k,\varepsilon} \to f_k$ as a generalization of (3.8) where the test function depend on ε.

To *prove* $f_{k,\varepsilon} \to f_k$, the weak convergence (3.7) does not suffice. This is so especially because the test function $\phi_\varepsilon(t,x)$ (which approximates the Dirac function) probes *small*-scales that are not compatible with the topology of the weak convergence (3.7). One possible proof is to establish a local version (3.7) that is commensurate with the scale of $\phi_\varepsilon(t,x)$. Doing so requires much technical effort. Instead, we *circumvent* this technical issue by testing $f_{k,\varepsilon}$ against a smooth test function g, after which the weak convergence (3.7) applies.

Proof. Given the chaos expansions in (3.2) and (3.6), it suffices to show $\widetilde{f}_k = f_k$. Since $u_\varepsilon(t,x) \to \mathscr{U}(t,x)$ in $L^2(\Omega)$, using the orthogonality of the chaos, i.e.,

$$\mathbb{E}\big[(u_\varepsilon - \mathscr{U})^2\big] = \sum_{k=0}^\infty \int_{\mathbb{R}^k_< \times \mathbb{R}^k} (f_{k,\varepsilon} - \widetilde{f}_k)^2 dsdy,$$

we see that, for each $k \geqslant 0$, $f_{k,\varepsilon} \to \widetilde{f}_k$ in $L^2(\mathbb{R}^k_< \times \mathbb{R}^k)$. Fix arbitrary $g \in C_c^\infty(\mathbb{R}^{2n})$, we consider

$$\langle f_{k,\varepsilon}, g\rangle := \int_{\mathbb{R}^k_< \times \mathbb{R}^k} f_{k,\varepsilon}(r_1,\ldots,r_k,y_1,\ldots,y_k) g(r_1,\ldots,r_k,y_1,\ldots,y_k) drdy. \tag{3.9}$$

To prove $\widetilde{f}_k = f_k$, it suffices to show $\langle f_{k,\varepsilon}, g\rangle \to \langle f_k, g\rangle$ as $\varepsilon \to 0$.

The formula (3.5) yields

$$\begin{aligned}\langle f_{k,\varepsilon}, g\rangle = \int_{\mathbb{R}^k_< \times \mathbb{R}^k} &\mathbb{E}_B\Big[u_0(x+B(t)) e^{\int_0^t\int_0^s R_\varepsilon(s-u,B(s)-B(u))duds - c_\varepsilon t} \\ &\Big(\int_{[0,t]^k} \prod_{i=1}^k \phi_\varepsilon(t-s_i-r_i, x+B(s_i)-y_i)ds\Big)\Big] g(r_1,\ldots,r_k,y_1,\ldots,y_k)drdy.\end{aligned} \tag{3.10}$$

With $\phi_\varepsilon(\bullet,\bullet) := \varepsilon^{-3}\phi(\varepsilon^{-2}\bullet, \varepsilon^{-1}\bullet)$, we perform a change of variables $r_i \mapsto \varepsilon^2 r_i' + t - s_i$, $y_i \mapsto \varepsilon y_i' + x + B(s_i)$ to rewrite the last expression as

$$\begin{aligned}\langle f_{k,\varepsilon}, g\rangle = \int_{\mathbb{R}^{3k}} \mathbf{1}_{A_\varepsilon \cap A'}(r',s) \prod_{i=1}^k \phi(-r_i', -y_i') \mathbb{E}_B\Big[u_0(x+B(t)) e^{\int_0^t\int_0^s R_\varepsilon(s-u,B(s)-B(u))duds - c_\varepsilon t} \\ \times g\big(\varepsilon^2 r_1' + t - s_1, \ldots, \varepsilon^2 r_k' + t - s_k, \varepsilon y_1' + x + B(s_1), \ldots, \varepsilon y_k' + x + B(s_k)\big)\Big] dr'dy'ds,\end{aligned}$$

where $A_\varepsilon := \{\varepsilon^2 r_1' + t - s_1 < \ldots < \varepsilon^2 r_k' + t - s_k\}$ and $A' := \{(s_1,\ldots,s_k) \in [0,t]^k\}$ translate the constraints on the old variables into the new ones. In order to pass to the limit, we note that, by Proposition 2.4, (2.3), (2.7) and our choice of c_ε, for any fixed $(s_1,\ldots,s_k) \in [0,t]^k$,

$$\Big(\int_0^t \int_0^s R_\varepsilon(s-u, B(s)-B(u))duds - c_\varepsilon t, B(s_1), \ldots, B(s_k), B(t)\Big) \Rightarrow \Big(\sigma_* W(t) - \tfrac{1}{2}\sigma_*^2 t, B(s_1), \ldots, B(s_k), B(t)\Big).$$

In addition, for any fixed $(r'_1,\ldots,r'_k)\in\mathbb{R}^k$, $\mathbf{1}_{A_\varepsilon\cap A'}(r',s)\to\mathbf{1}_{\{0<s_k<\ldots<s_1<t\}}$. By applying Proposition 2.2 and the dominated convergence theorem, we arrive at

$$\langle f_{k,\varepsilon},g\rangle \longrightarrow \int_{\mathbb{R}^{3k}} \mathbf{1}_{\{0<s_k<\ldots<s_1<t\}} \prod_{i=1}^{k}\phi(-r'_i,-y'_i)$$
$$\times\, \mathbb{E}_B\Big[u_0(x+B(t))e^{\sigma_* W(t)-\frac{1}{2}\sigma_*^2 t} g(t-s_1,\ldots,t-s_k,x+B(s_1),\ldots,x+B(s_k))\Big] dr'dy'ds.$$

In the last expression, integrate over (r_i,y_i) using $\int\phi drdy=1$, and perform a change of variables $t-s_i\mapsto s_i$. We see that it equals

$$\int_{\mathbb{R}^k_<\cap[0,t]^k}\mathbb{E}_B\Big[u_0(x+B(t))g(s_1,\ldots,s_k,x+B(t-s_1),\ldots,x+B(t-s_k))\Big]ds=\langle f_k,g\rangle\,.$$

The proof is complete. □

Remark 3.5. In the discrete setting, the convergence to SHE from the partition function of a random polymer with a weak randomness was proved in Alberts-Khanin-Quastel [AKQ14], also based on a chaos expansion.

Appendix A. Homogenization and stochasticity

In this appendix we investigate the behaviors of (1.3) at different scales. Let $u_0(x)\in C_b(\mathbb{R})$ be fixed as in the rest of the article. For $\alpha>0$, set

$$\xi_{\varepsilon,\alpha}(t,x)=\int_{\mathbb{R}^2}\phi_{\varepsilon,\alpha}(t-s,x-y)\xi(s,y)dyds,\quad \phi_{\varepsilon,\alpha}(t,x)=\varepsilon^{-\frac{3}{2}\alpha}\phi(\varepsilon^{-\alpha}t,\varepsilon^{-\frac{\alpha}{2}}x),$$

and consider the solution $v_{\varepsilon,\alpha}$ of

$$\partial_t v_{\varepsilon,\alpha}=\tfrac{1}{2}\partial_{xx}v_{\varepsilon,\alpha}+\varepsilon^{\frac{1}{2}-\frac{\alpha}{4}}\xi_{\varepsilon,\alpha}v_{\varepsilon,\alpha},\quad v_{\varepsilon,\alpha}(0,x)=u_0(x). \tag{A.1}$$

Referring to (1.2), we see that $\xi_{\varepsilon,2}=\xi_\varepsilon$. For $\alpha=2$, (A.1) is the same as (1.3) up to a centering by $-c_\varepsilon u_\varepsilon$. We thus view (A.1) as a generalization of (1.3) to scales $\alpha>0$. To see why the specific choice of prefactor $\varepsilon^{\frac{1}{2}-\frac{1}{4}\alpha}$ in (A.1) is relevant, perform a change of variable $v(t,x):=v_{\varepsilon,\alpha}(\varepsilon^\alpha t,\varepsilon^{\alpha/2}x)$ in (A.1) to bring the equation into the 'microscopic' coordinates. Using the scaling properties $\sqrt{AB}\,\xi(A\,\bullet,B\,\bullet)\overset{\text{law}}{=}\xi(\bullet,\bullet)$ of ξ, we see that v solves

$$\partial_t v=\tfrac{1}{2}\partial_{xx}v+\sqrt{\varepsilon}\eta(t,x)v, \tag{A.2}$$

where $\eta(t,x)=\int_{\mathbb{R}^2}\phi(t-s,x-y)\widetilde{\xi}(s,y)dyds$, for some $\widetilde{\xi}\overset{\text{law}}{=}\xi$. That is, the equation (A.1) encodes the behavior of (A.2) (which is α-independent) at the scale $(t,x)\sim(\varepsilon^{-\alpha},\varepsilon^{-\alpha/2})$.

Theorem 1.1 yields that $v_{\varepsilon,2}\exp(-c_\varepsilon t)\to\mathcal{U}$. On the other hand, a homogenization result was proved in [Bal11, PP12] at the scale $\alpha=1$, $v_{\varepsilon,1}(t,x)\to\bar{v}(t,x)\exp(c_* t)$, where $\bar{v}$ solves the unperturbed heat equation

$$\partial_t\bar{v}=\tfrac{1}{2}\partial_{xx}\bar{v},\quad \bar{v}(0,x)=u_0(x).$$

In the following, we establish an analogous homogenization result for $\alpha\in[1,2)$, together with a Gaussian fluctuation result.

Proposition A.1. *Fix $\alpha\in[1,2)$. For any given $(t,x)\in\mathbb{R}_+\times\mathbb{R}$, we have*

$$v_{\varepsilon,\alpha}(t,x)\exp(-\tfrac{c_* t}{\varepsilon^{\alpha-1}})\longrightarrow\bar{v}(t,x)\quad\textit{in probability}, \tag{A.3}$$

$$\mathbb{E}\Big[\Big|\varepsilon^{-\frac{2-\alpha}{4}}\big(v_{\varepsilon,\alpha}(t,x)-\mathbb{E}[v_{\varepsilon,\alpha}(t,x)]\big)\exp(-\tfrac{c_* t}{\varepsilon^{\alpha-1}})-\mathscr{V}(t,x)\Big|^2\Big]\longrightarrow 0. \tag{A.4}$$

where $\mathscr{V}$ solves the Edwards-Wilkinson equation

$$\partial_t\mathscr{V}=\tfrac{1}{2}\partial_{xx}\mathscr{V}+\bar{v}\xi,\quad \mathscr{V}(0,x)=0. \tag{A.5}$$

The result shows that if we start the microscopic dynamics (A.2) with the randomness of size $\sqrt{\varepsilon}$, then the small Gaussian fluctuations prevail in $t\sim\varepsilon^{-\alpha}$ for any $\alpha<2$. As we increase the time scale to $\alpha=2$, the random fluctuations become of order $O(1)$ and is described by the SHE.

Sketch of proof. Compared with Theorem 1.1 (i.e., $\alpha = 2$), the scale $\alpha \in [1,2)$ considered here is easier to analyze, so we only sketch the proof. Fix $\alpha \in [1,2)$ and set $\beta := \frac{1}{2} - \frac{\alpha}{4} > 0$. Apply the Stroock formula in a similar way as we established (3.3) and (3.5), here we have

$$v_\varepsilon(t,x)\exp(-\tfrac{c_* t}{\varepsilon^{\alpha-1}}) = \sum_{k=0}^{\infty} \varepsilon^{\beta k} I_k(\bar{f}_{k,\varepsilon^{\alpha/2}}), \tag{A.6}$$

where

$$\bar{f}_{k,\delta}(r_1,\ldots,r_k,y_1,\ldots,y_k) = \mathbb{E}_B\Big[u_0(x+B(t))\exp\Big(\varepsilon^{2\beta}X_\delta(t) + r_\varepsilon(t)\Big)\prod_{i=1}^{k}\Psi_{\delta,B}(r_i,y_i)\Big]. \tag{A.7}$$

With $\varepsilon^{\beta k} \leqslant \varepsilon^{\beta} \to 0$, for $k \geqslant 1$, it is not hard to show that

$$\sum_{k=1}^{\infty} \varepsilon^{\beta k} I_k(\bar{f}_{k,\varepsilon^{\alpha/2}}) \longrightarrow 0 \quad \text{in } L^2(\Omega),$$

and that $\bar{f}_{0,\varepsilon^{\alpha/2}} \to \mathbb{E}_B[u_0(x+B(t))] = \bar{v}(t,x)$. This concludes the first claim.

Moving onto the second claim regarding random fluctuations, we write

$$\varepsilon^{-\beta}(v_\varepsilon(t,x) - \mathbb{E}[v_\varepsilon(t,x)])\exp(-\tfrac{c_* t}{\varepsilon^{\alpha-1}}) = I_1(\bar{f}_{1,\varepsilon^{\alpha/2}}) + \sum_{k=2}^{\infty} \varepsilon^{\beta(k-1)} I_k(\bar{f}_{k,\varepsilon^{\alpha/2}}). \tag{A.8}$$

Again, With $\varepsilon^{\beta(k-1)} \leqslant \varepsilon^{\beta} \to 0$, for $k \geqslant 2$, it is not hard to show that the second term on the RHS goes to zero in $L^2(\Omega)$. This being the case, we focus on the first order chaos $I_1(\bar{f}_{1,\varepsilon^{\alpha/2}})$. Using a similar argument as in the proof of Lemma 3.3, here we have

$$\bar{f}_{1,\varepsilon^{\alpha/2}}(r,y) \to q(t-r,x-y)\int_{\mathbb{R}} u_0(z)q(r,y-z)dz = q(t-r,x-y)\bar{v}(r,y), \quad \text{in } L^2(\mathbb{R}^2),$$

which then yields

$$I_1(\bar{f}_{1,\varepsilon^{\alpha/2}}) \longrightarrow \int_0^t \int_{\mathbb{R}} q(t-r,x-y)\bar{v}(r,y)\xi(r,y)drdy, \quad \text{in } L^2(\Omega).$$

The expression $\int_0^t\int_{\mathbb{R}} q(t-r,x-y)\bar{v}(r,y)\xi(r,y)drdy$ is exactly $\mathscr{V}(t,x)$. We hence conclude the second claim. $\square$

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