

THE LIE COALGEBRA OF MULTIPLE POLYLOGARITHMS

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ABSTRACT. We use Goncharov’s coproduct of multiple polylogarithms to define a Lie coalgebra over an arbitrary field. It is generated by symbols subject to inductively defined relations, which we think of as functional relations for multiple polylogarithms. In particular, we have inversion relations and shuffle relations. We relate our definition to Goncharov’s Bloch groups, and to the concrete model for $\mathcal{L}(F)_{\leq 4}$ by Goncharov and Rudenko.

1. INTRODUCTION

1.1. The motivic Lie coalgebra. For a field F , one expects the existence of a graded Lie coalgebra $\mathcal{L}(F)$ such that the weight n part of its Chevalley-Eilenberg complex $\wedge^*(\mathcal{L}(F))$ computes the motivic cohomology groups $H_{\mathcal{M}}^i(F, \mathbb{Z}(n))$ (see e.g. [Gon94]). The existence of this so-called *motivic Lie coalgebra* is known for number fields [Gon05]. It is desirable to have a concrete description. Goncharov [Gon94] conjectures that $\wedge^*(\mathcal{L}(F))_n$ is rationally quasi-isomorphic to the Bloch complex $\Gamma(F, n)$ defined in [Gon95]. This complex has the form

$$(1) \quad \mathcal{B}_n(F) \xrightarrow{\delta} \cdots \xrightarrow{\delta} \mathcal{B}_{n-k}(F) \otimes \wedge^k(F^*) \xrightarrow{\delta} \cdots \xrightarrow{\delta} \mathcal{B}_2(F) \otimes \wedge^{n-2}(F^*) \xrightarrow{\delta} \wedge^n(F^*),$$

with each group $\mathcal{B}_k(F)$ being generated by symbols $\{x\}_k$ with $x \in F \cup \{\infty\}$ subject to relations that may be thought of as polylogarithm relations. The rightmost δ takes $\{x\}_{2 \otimes a}$ to $x \wedge (1-x) \wedge a$, and the others take $\{x\}_k \otimes a$ to $\{x\}_{k-1} \otimes x \wedge a$. Goncharov conjectures that $\mathcal{L}(F)_n = \mathcal{B}_n(F)$ for $n \leq 3$, but for $n = 4$, $\mathcal{L}(F)_4$ is larger. An explicit model for $\wedge^*(\mathcal{L}(F))_{n \leq 4}$ is given in Goncharov-Rudenko [GR18] (see Section 6 for a brief summary).

1.2. Multiple polylogarithms and Goncharov’s coproduct. The classical polylogarithms $\text{Li}_n(x)$ have multivariable generalizations $\text{Li}_{n_1, \dots, n_d}(x_1, \dots, x_d)$ called *multiple polylogarithms*. Their properties are extensively studied by Goncharov [Gon01, Gon05]. He shows that one may view $\text{Li}_{n_1, \dots, n_d}(x_1, \dots, x_d)$ and $\log(x)$ as elements $\text{Li}_{n_1, \dots, n_d}^{\mathcal{C}}(x_1, \dots, x_d)$ and $\log^{\mathcal{C}}(x)$ of a certain Hopf algebra (the Hopf algebra of framed rational Hodge structures [Gon05, appendix]). In fact, they generate a Hopf subalgebra of this Hopf algebra, and Goncharov gives an explicit formula for the coproduct. It is best described using the generating series

$$(2) \quad \begin{aligned} \text{Li}^{\mathcal{C}}(x_1, \dots, x_d | t_1, \dots, t_d) &= \sum_{n_i \geq 1} \text{Li}_{n_1, \dots, n_d}^{\mathcal{C}}(x_1, \dots, x_d) t_1^{n_1-1} \cdots t_d^{n_d-1}, \\ x^t &= \sum_{n \geq 0} \frac{1}{n!} \log^{\mathcal{C}}(x)^n t^n. \end{aligned}$$

The coproduct of a generating series is defined termwise (fixing the t_i).

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Theorem 1.1 ([Gon05, Prop. 6.1]). *The coproduct Δ of $\text{Li}^{\mathcal{C}}(x_1, \dots, x_d | t_1, \dots, t_d)$ is given by*

$$(3) \quad \sum \text{Li}^{\mathcal{C}}(x_{i_1 \rightarrow i_2}, \dots, x_{i_k \rightarrow i_{k+1}} | t_{j_1}, \dots, t_{j_k}) \otimes \prod_{\alpha=0}^k (-1)^{j_\alpha - i_\alpha} x_{i_\alpha \rightarrow i_{\alpha+1}}^{t_{j_\alpha}} \left(\text{Li}^{\mathcal{C}}(x_{j_\alpha-1}^{-1}, x_{j_\alpha-2}^{-1}, \dots, x_{i_\alpha}^{-1} | t_{j_\alpha} - t_{j_\alpha-1}, \dots, t_{j_\alpha} - t_{i_\alpha}) \cdot \text{Li}^{\mathcal{C}}(x_{j_\alpha+1}, x_{j_\alpha+2}, \dots, x_{i_{\alpha+1}-1} | t_{j_\alpha+1} - t_{j_\alpha}, \dots, t_{i_{\alpha+1}-1} - t_{j_\alpha}) \right)$$

The sum is over all $k = 0, \dots, d$, and all sequences $\{i_\alpha\}_{\alpha=0}^{k+1}$ and $\{j_\alpha\}_{\alpha=0}^k$ with

$$(4) \quad i_\alpha \leq j_\alpha < i_{\alpha+1}, \quad j_0 = i_0 = 0, \quad i_{k+1} = d + 1,$$

and by definition we have $x_{i \rightarrow j} = \prod_{r=i}^{j-1} x_r$ (note that this convention differs from Goncharov's, but works better in this paper) and $\text{Li}^{\mathcal{C}}(\emptyset | \emptyset) = 1$.

Example 1.2. When $d = 2$, the $\{i_\alpha\}$ and $\{j_\alpha\}$ satisfying (4) are:

$$(5) \quad \begin{matrix} i_0 & j_0 & i_1 & j_1 & i_2 & j_2 & i_3 \\ (0, 0 | 1, 1 | 2, 2 | 3), & (0, 0 | 1, 1 | 3), & (0, 0 | 1, 2 | 3), & (0, 0 | 2, 2 | 3), & (0, 0 | 3), \end{matrix}$$

and it follows that

$$(6) \quad \Delta \text{Li}^{\mathcal{C}}(x_1, x_2 | t_1, t_2) = \text{Li}^{\mathcal{C}}(x_1, x_2 | t_1, t_2) \otimes x_1^{t_1} x_2^{t_2} + \text{Li}^{\mathcal{C}}(x_1 x_2 | t_1) \otimes (x_1 x_2)^{t_1} \text{Li}^{\mathcal{C}}(x_2 | t_2 - t_1) - \text{Li}^{\mathcal{C}}(x_1 x_2 | t_2) \otimes (x_1 x_2)^{t_2} \text{Li}^{\mathcal{C}}(x_1^{-1} | t_2 - t_1) + \text{Li}^{\mathcal{C}}(x_2 | t_2) \otimes \text{Li}^{\mathcal{C}}(x_1 | t_1) x_2^{t_2} + 1 \otimes \text{Li}^{\mathcal{C}}(x_1, x_2 | t_1, t_2).$$

One can then compute $\Delta \text{Li}_{n_1, n_2}^{\mathcal{C}}(x_1, x_2)$ as the coefficient of $t_1^{n_1-1} t_2^{n_2-1}$ of the right-hand side of (6).

Given a graded Hopf algebra H , we have a Lie coalgebra $L = \frac{H_{>0}}{H_{>0}H_{>0}}$ with cobracket induced by the coproduct. We are thus interested in the quotient by products. The following elementary corollary of Theorem 1.1 is our main motivation.

Corollary 1.3. Modulo products and constants, $\Delta \text{Li}^{\mathcal{C}}(x_1, \dots, x_d | t_1, \dots, t_d)$ can be written as

$$(7) \quad \begin{aligned} & \text{Li}^{\mathcal{C}}(\mathbf{x}_{1, \dots, d} | \mathbf{t}_{1, \dots, d}) \otimes \sum_{p=1}^d t_p \log^{\mathcal{C}}(x_p) + \\ & \sum_{p=2}^d \text{Li}^{\mathcal{C}}(\mathbf{x}_{p, \dots, d} | \mathbf{t}_{p, \dots, d}) \otimes \text{Li}^{\mathcal{C}}(\mathbf{x}_{1, \dots, p-1} | \mathbf{t}_{1, \dots, p-1}) + \\ & \sum_{1 \leq p < q \leq d} \text{Li}^{\mathcal{C}}(\mathbf{x}_{1, \dots, p \rightarrow q, \dots, d} | \mathbf{t}_{1, \dots, p, q+1, \dots, d}) \otimes \text{Li}^{\mathcal{C}}(\mathbf{x}_{p+1, \dots, q} | \mathbf{t}_{p+1, \dots, q} - t_p) + \\ & \sum_{1 \leq p < q \leq d} (-1)^{q-p} \text{Li}^{\mathcal{C}}(\mathbf{x}_{1, \dots, p \rightarrow q, \dots, d} | \mathbf{t}_{1, \dots, p-1, q, \dots, d}) \otimes \text{Li}^{\mathcal{C}}(x_{q-1}^{-1}, x_{q-2}^{-1}, \dots, x_p^{-1} | t_q - \mathbf{t}_{q-1, \dots, p}) \end{aligned}$$

Where $\mathbf{x}_{1, \dots, p \rightarrow q, \dots, d}$ is shorthand for $(x_1, \dots, x_{p-1}, \prod_{r=p}^q x_r, x_{q+1}, \dots, x_d)$ and dots indicate that indices increase (or decrease) by 1. We stress that the product is from p to q , not to $q-1$.

Proof. One easily checks that the only sequences $\{i_\alpha\}$ and $\{j_\alpha\}$ for which the corresponding term in (3) does not involve products are

$$(8) \quad \begin{aligned} & (0, 0 | 1, 1 | 2, 2 | \cdots | d, d |, d+1) \\ & (0, 0 | \overset{i_0}{p}, \overset{j_0}{p} | \overset{i_1}{p+1}, \overset{j_1}{p+1} | \cdots | \overset{i_k}{d}, \overset{j_k}{d} |, \overset{i_{k+1}}{d+1}) \\ & (0, 0 | 1, 1 | \cdots | \overset{i_p}{p}, \overset{j_p}{p} | \overset{i_{p+1}}{q+1}, \overset{j_{p+1}}{q+1} | \cdots | \overset{i_k}{d}, \overset{j_k}{d} |, \overset{i_{k+1}}{d+1}) \\ & (0, 0 | 1, 1 | \cdots | \overset{i_p}{p}, \overset{j_p}{q} | \overset{i_{p+1}}{q+1}, \overset{j_{p+1}}{q+1} | \cdots | \overset{i_k}{d}, \overset{j_k}{d} |, \overset{i_{k+1}}{d+1}), \end{aligned}$$

and that the corresponding terms are exactly those given in (7). \square

1.3. Real single valued polylogarithms. One has single valued variants $\mathcal{L}_n(x)$ of the classical polylogarithms (see e.g. [DB12, Zag91])

$$(9) \quad \mathcal{L}_n(z) = \begin{cases} \text{Re if } n \text{ is odd} \\ \text{Im if } n \text{ is even} \end{cases} \left(\sum_{r=0}^{n-1} \frac{2^r B_r}{r!} \text{Li}_{n-r}(z) (\log |z|)^r \right).$$

The following result justifies thinking about the relations in $\mathcal{B}_n(F)$ as polylogarithm relations:

Theorem 1.4 ([Zag91, Prop. 3]). *For any element $\alpha = \sum n_i f_i(t) \in \mathcal{B}_n(\mathbb{C}(t))$ with $\delta(\alpha) = 0$ in $\mathcal{B}_{n-1}(\mathbb{C}(t)) \otimes \mathbb{C}(t)^*$ we have*

$$(10) \quad \sum n_i \mathcal{L}_n(f_i(t)) = \text{constant}.$$

There are also single valued analogues $\mathcal{L}_{n_1, \dots, n_d}$ of the multiple polylogarithms $\text{Li}_{n_1, \dots, n_d}$. There is no closed formula, but they can be computed from a variation matrix [Zha02]. We shall here only need that $\mathcal{L}_{n_1, \dots, n_d}$ is defined on

$$(11) \quad S_d(\mathbb{C}) = \left\{ (x_1, \dots, x_d) \in (\mathbb{C}^*)^d \mid \prod_{r=i}^j x_r \neq 1 \text{ for all } i \leq j \in \{1, \dots, d\} \right\}$$

and that

$$(12) \quad \lim_{x_i \rightarrow 0} \mathcal{L}_{n_1, \dots, n_d}(x_1, \dots, x_d) = 0 \quad \text{for all } i.$$

The functions $\mathcal{L}_n(x)$ also satisfy that $\lim_{x \rightarrow \infty} \mathcal{L}_n(x) = 0$ for $n > 1$, but in higher depth the limit as x_i tends to ∞ is no longer 0.

1.4. Structure of the paper. In Section 2 we define a purely symbolic coalgebra $\mathbb{L}^{\text{symb}}(F)$ with no relations. We have $\mathbb{L}_1^{\text{symb}}(F) = F^*$, and $\mathbb{L}_{>1}^{\text{symb}}$ is generated by symbols $[x_1, \dots, x_d]_{n_1, \dots, n_d}$ with $(x_1, \dots, x_d) \in S_d(F)$, where d and n_1, \dots, n_d are positive integers. Thinking of a symbol as a polylogarithm we define the cobracket as in Corollary 1.3 and show directly that $\delta^2 = 0$. In Section 3 we inductively define a group of relations $R_n(F)$ and define

$$(13) \quad \mathbb{L}_n(F) = \mathbb{L}_n^{\text{symb}}(F) / R_n(F).$$

The definition mimics Goncharov's definition of relations in $\mathcal{B}_n(F)$, and the proof that the cobracket takes relations to 0 in $\wedge^2(\mathbb{L}(F))$ follows Goncharov as well. Section 3.1 gives some basic examples of relations, and Section 3.2 discusses the problem of defining symbols when $(x_1, \dots, x_d) \notin S_d(F)$. For example, $[1, 1]_{1,1}$ is not well defined. In Section 3.3 we speculate that $\mathbb{L}(F)$ is the motivic Lie coalgebra $\mathcal{L}(F)$ and conjecture a generalization of Theorem 1.4, which justifies thinking of $R_n(F)$ as polylogarithm relations. Section 4 discusses the *inversion relations*, which are inspired by Goncharov's inversion relations for multiple polylogarithms. In

particular, we show that in $\mathbb{L}(F)$ one can express each symbol $[x_d^{-1}, \dots, x_1^{-1}]_{n_d, \dots, n_1}$ in terms of symbols involving only non-inverted x_i . Section 4.1 shows that one can use this to define an alternative cobracket on $\mathbb{L}^{\text{symb}}(F)$ without inverted x_i . The relations are the same. The alternative one may in fact be more natural (see e.g. Remark 5.3). Section 5 briefly discusses the shuffle product relations showing that at least in low depth a shuffle product is 0 in $\mathbb{L}(F)$. Finally, Section 6 relates our work to that of Goncharov and Rudenko, who gave a concrete model for $\mathcal{L}_{n \leq 4}(F)$.

Remark 1.5. Although our work is heavily inspired by the work of Goncharov, it does not require any of Goncharov's results, except for motivation.

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2. A PURELY SYMBOLIC LIE COALGEBRA

For a field F , and a positive integer d , let

$$(14) \quad S_d(F) = \left\{ (x_1, \dots, x_d) \in (F^*)^d \mid \prod_{r=i}^j x_r \neq 1 \text{ for all } i \leq j \in \{1, \dots, d\} \right\}.$$

We wish to define a graded Lie coalgebra

$$(15) \quad \mathbb{L}^{\text{symb}}(F) = \bigoplus_{n=1}^{\infty} \mathbb{L}_n^{\text{symb}}(F).$$

We first define $\mathbb{L}_1^{\text{symb}}(F) = F^*$, which we shall identify with the abelian group generated by symbols $[x]_1$ with $x \in F \setminus \{0, 1\}$ and $[x]_0$ for $x \in F \setminus \{0, 1\}$ subject to the relations

$$(16) \quad [x]_1 = -[1-x]_0, \quad [xy]_0 = [x]_0 + [y]_0.$$

For $n > 1$, define $\mathbb{L}_n^{\text{symb}}(F)$ to be generated by symbols $[x_1, \dots, x_d]_{n_1, \dots, n_d}$ with

$$(17) \quad (x_1, \dots, x_d) \in S_d(F), \quad d, n_1, \dots, n_d \in \mathbb{Z}_+, \quad n_1 + \dots + n_d = n.$$

We refer to $n_1 + \dots + n_d$ as the *weight* and d as the *depth* of a symbol, but stress that $[x]_0$ is in weight 1, not 0. We think of $[x_1, \dots, x_d]_{n_1, \dots, n_d}$ as representing $\text{Li}_{n_1, \dots, n_d}(x_1, \dots, x_d)$ and $[x]_0$ as representing $\log(x)$.

2.1. The cobracket on $\mathbb{L}^{\text{symb}}(F)$. As in (2) we define

$$(18) \quad [x_1, \dots, x_d | t_1, \dots, t_d] = \sum_{n_i \geq 1} [x_1, \dots, x_d]_{n_1, \dots, n_d} t_1^{n_1-1} \dots t_d^{n_d-1},$$

and define $\delta: \mathbb{L}^{\text{symb}}(F) \rightarrow \wedge^2(\mathbb{L}^{\text{symb}}(F))$ to be zero in weight 1, and otherwise given by (7). It is convenient to write $\delta = \delta_1 + \delta_2 + \delta_3 + \delta_4$, where for $X = [\mathbf{x}_1, \dots, d | \mathbf{t}_1, \dots, d]$

$$\begin{aligned}
(19) \quad \delta_1(X) &= X \wedge \left(\sum_{p=1}^d t_p [x_p]_0 \right) \\
\delta_2(X) &= \sum_{p=2}^d [\mathbf{x}_p, \dots, d | \mathbf{t}_p, \dots, d] \wedge [\mathbf{x}_1, \dots, p-1 | \mathbf{t}_1, \dots, p-1] \\
\delta_3(X) &= \sum_{1 \leq p < q \leq d} [\mathbf{x}_1, \dots, p \rightarrow q, \dots, d | \mathbf{t}_1, \dots, p, q+1, \dots, d] \wedge [\mathbf{x}_{p+1}, \dots, q | \mathbf{t}_{p+1}, \dots, q - t_p] \\
\delta_4(X) &= \sum_{1 \leq p < q \leq d} (-1)^{q-p} [\mathbf{x}_1, \dots, p \rightarrow q, \dots, d | \mathbf{t}_1, \dots, p-1, q, \dots, d] \wedge [x_{q-1}^{-1}, \dots, x_p^{-1} | t_q - \mathbf{t}_{q-1}, \dots, p].
\end{aligned}$$

Example 2.1. In depth 1, we have $\delta[x|t] = [x|t] \wedge t[x]_0$, from which it follows that $\delta[x]_n = [x]_{n-1} \wedge [x]_0$ for $n > 1$. Note the case $\delta[x]_2 = [x]_1 \wedge [x]_0 = x \wedge (1-x) \in \wedge^2(F^*)$. We thus recover the boundary map in the Bloch complex.

Example 2.2. In depth 2, $\delta[x_1, x_2 | t_1, t_2]$ equals (compare with (6))

$$(20) \quad \delta[x_1, x_2 | t_1, t_2] = [x_1, x_2 | t_1, t_2] \wedge (t_1[x_1]_0 + t_2[x_2]_0) + [x_2 | t_2] \wedge [x_1 | t_1] + [x_1 x_2 | t_1] \wedge [x_2 | t_2 - t_1] - [x_1 x_2 | t_2] \wedge [x_1^{-1} | t_2 - t_1].$$

In particular, $\delta[x_1, x_2]_{r,s}$ is given by

$$(21) \quad \begin{aligned}
& [x_1, x_2]_{r-1,s} \wedge [x_1]_0 + [x_1, x_2]_{r,s-1} \wedge [x_2]_0 + [x_2]_s \wedge [x_1]_r \\
& + \sum_{i=1}^r (-1)^{r-i} \binom{r+s-1-i}{s-1} [x_1 x_2]_i \wedge [x_2]_{r+s-i} \\
& + \sum_{i=1}^s (-1)^r \binom{r+s-1-i}{r-1} [x_1 x_2]_i \wedge [x_1^{-1}]_{r+s-i}.
\end{aligned}$$

In the case where r or s is 1, the symbols $[x_1, x_2]_{0,s}$ and $[x_1, x_2]_{r,0}$ are interpreted as 0.

Remark 2.3. In our definition of $\wedge^2(\mathbb{L}^{\text{symb}}(F))$, $x \wedge x$ is identically 0, not 2-torsion.

Theorem 2.4. *The cobracket above makes $\mathbb{L}^{\text{symb}}(F)$ into a Lie coalgebra.*

Proof. We must show that $\delta^2 = 0$. To do this, it is enough to show that

$$(22) \quad \begin{aligned}
& \delta_1^2 = 0, \quad \delta_2^2 = 0, \quad \delta_1 \delta_i + \delta_i \delta_1 = 0, \\
& \delta_3^2 + \delta_2 \delta_3 + \delta_3 \delta_2 = 0, \quad \delta_2 \delta_4 + \delta_4 \delta_2 + \delta_4^2 + \delta_3 \delta_4 + \delta_4 \delta_3 = 0.
\end{aligned}$$

The proof that $\delta_1^2 = 0$ is elementary. The remaining equalities are all straightforward, so for brevity, we prove only that $\delta_2^2 = 0$, and that $\delta_1 \delta_4 + \delta_4 \delta_1 = 0$. Firstly, $\delta_2^2([\mathbf{x}_1, \dots, d | \mathbf{t}_1, \dots, d])$ is given by

$$(23) \quad \begin{aligned}
& \sum_{2 \leq p \leq d} \delta_2[\mathbf{x}_p, \dots, d | \mathbf{t}_p, \dots, d] \wedge [\mathbf{x}_1, \dots, p-1 | \mathbf{t}_1, \dots, p-1] - \sum_{2 \leq p \leq d} [\mathbf{x}_p, \dots, d | \mathbf{t}_p, \dots, d] \wedge \delta_2[\mathbf{x}_1, \dots, p-1 | \mathbf{t}_1, \dots, p-1] \\
& = \sum_{2 \leq p \leq d} \sum_{p+1 \leq r \leq d} [\mathbf{x}_r, \dots, d | \mathbf{t}_r, \dots, d] \wedge [\mathbf{x}_p, \dots, r-1 | \mathbf{t}_p, \dots, r-1] \wedge [\mathbf{x}_1, \dots, p-1 | \mathbf{t}_1, \dots, p-1] \\
& - \sum_{2 \leq p \leq d} \sum_{2 \leq r \leq p-1} [\mathbf{x}_p, \dots, d | \mathbf{t}_p, \dots, d] \wedge [\mathbf{x}_r, \dots, p-1 | \mathbf{t}_r, \dots, p-1] \wedge [\mathbf{x}_1, \dots, r-1 | \mathbf{t}_1, \dots, r-1] = 0.
\end{aligned}$$

Similarly, $\delta_1 \delta_4([\mathbf{x}_1, \dots, d | \mathbf{t}_1, \dots, d])$ equals

$$(24) \quad \sum_{1 \leq p < q \leq d} (-1)^{q-p} \delta_1[\mathbf{x}_1, \dots, p \rightarrow q, \dots, d | \mathbf{t}_1, \dots, p-1, q, \dots, d] \wedge [x_{q-1}^{-1}, \dots, x_p^{-1} | t_q - \mathbf{t}_{q-1, \dots, p}] \\ - (-1)^{q-p} [\mathbf{x}_1, \dots, p \rightarrow q, \dots, d | \mathbf{t}_1, \dots, p-1, q, \dots, d] \wedge \delta_1[x_{q-1}^{-1}, \dots, x_p^{-1} | t_q - \mathbf{t}_{q-1, \dots, p}],$$

which equals

$$(25) \quad = \sum_{1 \leq p < q \leq d} (-1)^{q-p} [\mathbf{x}_1, \dots, p \rightarrow q, \dots, d | \mathbf{t}_1, \dots, p-1, q, \dots, d] \wedge \\ \left(\sum_{1 \leq r \leq p-1} t_r [x_r]_0 + t_q [x_p \cdots x_q]_0 + \sum_{q+1 \leq r \leq d} t_r [x_r]_0 \right) \wedge [x_{q-1}^{-1}, \dots, x_p^{-1} | t_q - \mathbf{t}_{q-1, \dots, p}] \\ - (-1)^{q-p} [\mathbf{x}_1, \dots, p \rightarrow q, \dots, d | \mathbf{t}_1, \dots, p-1, q, \dots, d] \wedge [x_{q-1}^{-1}, \dots, x_p^{-1} | t_q - \mathbf{t}_{q-1, \dots, p}] \wedge \left(\sum_{p \leq r \leq q-1} (t_q - t_r) [x_r^{-1}]_0 \right) \\ = - \sum_{1 \leq p < q \leq d} (-1)^{q-p} [\mathbf{x}_1, \dots, p \rightarrow q, \dots, d | \mathbf{t}_1, \dots, p-1, q, \dots, d] \wedge [x_{q-1}^{-1}, \dots, x_p^{-1} | t_q - \mathbf{t}_{q-1, \dots, p}] \wedge \left(\sum_{1 \leq r \leq d} t_r [x_r]_0 \right).$$

Finally,

$$(26) \quad \delta_4 \delta_1([\mathbf{x}_1, \dots, d | \mathbf{t}_1, \dots, d]) = \delta_4[\mathbf{x}_1, \dots, d | \mathbf{t}_1, \dots, d] \wedge \left(\sum_{p=1}^d t_p [x_p]_0 \right) = \\ = \sum_{1 \leq p < q \leq d} (-1)^{q-p} [\mathbf{x}_1, \dots, p \rightarrow q, \dots, d | \mathbf{t}_1, \dots, p-1, q, \dots, d] \wedge [x_{q-1}^{-1}, \dots, x_p^{-1} | t_q - \mathbf{t}_{q-1, \dots, p}] \wedge \left(\sum_{p=1}^d t_p [x_p]_0 \right),$$

and it follows that $\delta_1 \delta_4 + \delta_4 \delta_1 = 0$. \square

Remark 2.5. Theorem 2.4 also holds without requiring that the tuples are in $S_d(F)$ (same proof). The problem with arbitrary tuples arises when defining the relations; see Section 3.2.

2.2. Terms involving zero or infinity. We shall also define elements $[x_1, \dots, x_d]_{n_1, \dots, n_d}$ when some of the x_i are 0 or ∞ , but we still require that all consecutive products $\prod_{r=i}^j x_r$ are well defined and not 1 ($\infty x_i = \infty \neq 1$; 0∞ is undefined). When some x_i are 0, $[x_1, \dots, x_d]_{n_1, \dots, n_d}$ is defined to be zero (this is motivated by (12)). When some x_i are ∞ the definition is more subtle and we refer to Section 4. All we need for now is that $[\infty]_n = 0$ for $n > 1$ (and undefined for $n=1$).

3. THE RELATIONS

We now define groups $R_n(F)$ of relations in $\mathbb{L}_n^{\text{symb}}(F)$. We can then define

$$(27) \quad \mathbb{L}_n(F) = \mathbb{L}_n^{\text{symb}}(F) / R_n(F).$$

The definition is inductive starting with the definition of $R_1(F)$ to be the trivial group, so that $\mathbb{L}_1(F) = F^*$. Suppose $n > 1$ and that $\mathbb{L}_k(K)$ has been defined for all fields K and all $k < n$. Then

$$(28) \quad \wedge^2 (\mathbb{L}(K))_n = \bigoplus_{k+l=n} \mathbb{L}_k(K) \wedge \mathbb{L}_l(K)$$

is also well defined for all K . Let

$$(29) \quad \mathcal{A}_n(K) = \text{Ker} \left(\delta: \mathbb{L}_n^{\text{sy mb}}(K) \rightarrow \wedge^2(\mathbb{L}(K))_n \right)$$

Mimicking Goncharov's definition of relations in the Bloch complexes [Gon95, p. 221] we wish to define $R_n(F) \subset \mathbb{L}_n^{\text{sy mb}}(F)$ to be generated by elements $\alpha(p) - \alpha(q)$, where p and q are points on a connected (geometrically irreducible) smooth curve X over F with function field $F(X)$ and α is an element in $\mathcal{A}_n(F(X))$. The only problem with this is our requirement that all tuples be well defined (allowing for 0 and ∞ ; see Section 2.2).

Each element $\alpha \in \mathbb{L}^{\text{sy mb}}(F(X))$ is a linear combination of terms $[f_1, \dots, f_d]_{n_1, \dots, n_d}$, which we refer to as *terms of α* .

Definition 3.1. We say that $\alpha \in \mathbb{L}^{\text{sy mb}}(F(X))$ is *well defined* at $x \in X$ if all products $\prod_{r=i}^j f_r(x)$ are defined and distinct from 1, for each term $[f_1, \dots, f_d]_{n_1, \dots, n_d}$ of α .

By induction, we see that if $\alpha \in \mathcal{A}_n(F(X))$, we can rearrange the terms $\delta(\alpha)$ as a linear combination of terms $(\beta(q) - \beta(q')) \otimes y$, where Y is a smooth connected curve over $F(X)$, q and q' are points in Y , $\beta \in \mathcal{A}_k(F(X)(Y))$ and $y \in \mathbb{L}_{n-k}^{\text{sy mb}}(F(X))$.

Definition 3.2. All $\alpha \in \mathcal{A}_2$ are *proper*. For $n > 2$, $\alpha \in \mathcal{A}_n(F(X))$ is *proper* if whenever α is well defined at $x \in X$, there is an arrangement of $\delta(\alpha)$ as above with $\beta(q)$, $\beta(q')$, and y all well defined at x (no cancelation of undefined terms) with β proper.

Definition 3.3. The group $R_n(F)$ is generated by elements of the form $\alpha(p) - \alpha(q)$, where $\alpha \in \mathcal{A}_n(F(X))$ is proper and well defined at $p, q \in X$.

We now prove that $\mathbb{L}(F)$ is also a Lie coalgebra. To do this we must prove that the cobracket respects the relations. The proof is similar to [Gon95, Lemma 1.16].

Theorem 3.4. *The map $\delta: \mathbb{L}_n^{\text{sy mb}}(F) \rightarrow \wedge^2(\mathbb{L}(F))_n$ takes $R_n(F)$ to 0.*

Proof. It is enough to show that for any proper $\alpha \in \mathcal{A}_n(F(X))$, the element $\delta(\alpha(p))$ is zero in $\wedge^2(\mathbb{L}(F))_n$ for all $p \in X$ where $\alpha(p)$ is defined. Fix such X , α , and p and write $\delta(\alpha)$ in $\wedge^2(\mathbb{L}^{\text{sy mb}}(F(X)))$ as a linear combination of terms $(\beta(q) - \beta(q')) \wedge y$ as above. It follows that $\delta(\alpha(p))$ is a linear combination of elements of the form $(\beta(q)(p) - \beta(q')(p)) \otimes y(p)$, and the result follows by showing that $\beta(q)(p) - \beta(q')(p)$ is in $R_k(F)$. Let β_p be the element in $\mathbb{L}_k^{\text{sy mb}}(F(Y(p)))$ obtained from β by restriction to the fiber $Y(p)$ over p . Since β is in $\mathcal{A}_k(F(X)(Y))$, an induction argument shows that β_p is in $\mathcal{A}_k(F(Y(p)))$ and is proper. Note that q and q' can be regarded as maps $X \rightarrow Y$, so restriction to $p \in X$ determines points q_p and q'_p in $Y(p)$. We then have

$$(30) \quad \beta(q)(p) - \beta(q')(p) = \beta_p(q_p) - \beta_p(q'_p) \in R_k(F).$$

This concludes the proof. \square

3.1. Basic examples of relations. In the following examples we repeatedly use the fact that for $\alpha \in \mathbb{L}^{\text{sy mb}}(F(x))$, $\delta(\alpha) = 0 \in \wedge^2(\mathbb{L}(F(x)))$ implies that $\alpha(x)$ is constant in $\mathbb{L}(F)$.

Example 3.5. Let $\alpha_n = 2([x]_n + (-1)^n [x^{-1}]_n) \in \mathbb{L}_n^{\text{sy mb}}(F)$. By induction, $\alpha_n \in \mathcal{A}_n(F(x))$ when $n \geq 2$, so $\alpha_n(x)$ is constant in $\mathbb{L}_n(F)$. Hence, $\alpha_n(x) = \alpha_n(0) = 2[0]_n + 2(-1)^n [\infty]_n = 0$. Note, however, that $\alpha_1(x) = -2[x]_0$.

Example 3.6. One easily shows that $\alpha = [x, y]_{1,1} + [x]_2 - \left[\frac{x(1-y)}{1-xy} \right]_2$ is in $\mathcal{A}_2(F(x, y))$. Considering the specialization at $(x, y) = (0, y)$ it follows that $[x, y]_{1,1} + [x]_2 - \left[\frac{x(1-y)}{1-xy} \right]_2 = 0 \in \mathbb{L}_2(F)$ whenever all terms are defined. This means that every element in $\mathbb{L}_2(F)$ can be expressed using only terms in depth 1.

Example 3.7 (Five term relation). Similarly, we obtain that

$$(31) \quad [x]_2 + [y]_2 - [xy]_2 - \left[\frac{y(1-x)}{1-xy} \right]_2 - \left[\frac{x(1-y)}{1-xy} \right]_2 = 0 \in \mathbb{L}_2(F).$$

Proposition 3.8. Modulo 2-torsion every element in $\mathbb{L}_3(F)$ can be written in terms of depth 1 symbols.

Proof. A simple computation shows that the elements

$$(32) \quad [x, y]_{2,1} + [x]_3 + \left[\frac{1-y}{1-xy} \right]_3 + [xy]_3 + \left[-\frac{xy}{1-xy} \right]_3 - [1-y]_3 - \left[\frac{x(1-y)}{1-xy} \right]_3$$

and

$$(33) \quad [x, y, z]_{1,1,1} - \left[-\frac{y}{1-y} \right]_3 + \left[\frac{1-x}{1-xyz} \right]_3 - [xy]_3 + \left[\frac{xy(1-z)}{1-xyz} \right]_3 \\ - [1-x]_3 + \left[-\frac{y(1-x)}{1-y} \right]_3 + \left[-\frac{y(1-z)}{1-y} \right]_3 - \left[-\frac{y(1-x)(1-z)}{(1-xyz)(1-y)} \right]_3$$

have vanishing cobracket modulo 2-torsion. Specializing at $(0, y)$ and $(x, 0, z)$, respectively, proves the result for $[x, y]_{2,1}$ and $[x, y, z]_{1,1,1}$. Finally, $[x, y]_{1,2} + [y, x]_{2,1} + [xy]_3$ also has vanishing cobracket, concluding the proof. \square

Remark 3.9. With more work one can show that (up to torsion) every element in $\mathbb{L}_4(F)$ can be expressed in terms of terms of the form $[x]_4$ and $[x, y]_{3,1}$. For example, one has

$$(34) \quad [x, y]_{2,2} = [y]_4 + [xy]_4 + [y, x]_{3,1} + [xy, x^{-1}]_{3,1} - [x, y]_{3,1}.$$

Similar equations for $[x, y, z]_{2,1,1}$, $[x, y, z]_{1,2,1}$, $[x, y, z]_{1,1,2}$, and $[x, y, z, w]_{1,1,1,1}$ are more complicated, and we omit them.

3.2. The issue of arbitrary symbols. The following argument shows that not all symbols can be meaningfully defined. By Example 3.6 we should have for any a

$$(35) \quad \left[x, \frac{x-a}{x-ax} \right]_{1,1} + [x]_2 - [a]_2 = 0.$$

Setting $x = 1$, we would thus get that $[1, 1]_{1,1} + [1]_2 = [a]_2$ for any a , which would imply that $\mathbb{L}_2(F) = 0$.

Remark 3.10. We believe that the reason for this is that $\mathcal{L}_{1,1}(x, y)$ does not have a limit as x and y tend to 1. One has similar issues with symbols such as $[x_1, 0, \infty]_{1,1,1}$.

Remark 3.11. One can give meaning to some additional symbols, but these should always be expressible in terms of symbols in $S_d(F)$. For example, one may define $[x, x^{-1}]_{1,1} = -[x]_2$. We shall not pursue this here.

3.3. Conjectures and speculation. Conjecture 3.12 below is a natural generalization of Theorem 1.4. Consider the map

$$(36) \quad r: \mathbb{L}^{\text{symb}}(\mathbb{C}) \rightarrow \mathbb{R}, \quad [x_1, \dots, x_d]_{n_1, \dots, n_d} \rightarrow \mathcal{L}_{n_1, \dots, n_d}(x_1, \dots, x_d), \quad [x]_0 \mapsto \log(|x|).$$

Conjecture 3.12. If $\alpha \in \mathbb{L}(\mathbb{C}(t))$ is such that $\delta(\alpha) = 0$, then $r(\alpha(t)) = 0$ is constant in t .

Remark 3.13. We have verified Conjecture 3.12 numerically for many examples in weight 4 and lower.

We also speculate more boldly that $\mathbb{L}(F)$ is rationally isomorphic to the motivic Lie coalgebra, so that the weight n part of the complex

$$(37) \quad \mathbb{L}(F) \xrightarrow{\delta} \wedge^2(\mathbb{L}(F)) \xrightarrow{\delta \wedge \text{id} - \text{id} \wedge \delta} \wedge^3(\mathbb{L}(F)) \longrightarrow \dots$$

should compute motivic cohomology, rationally. For a finite field F_q with q elements the only motivic cohomology groups are

$$(38) \quad H_{\mathcal{M}}^1(F_q, \mathbb{Z}(n)) = K_{2n-1}(F_q) = \mathbb{Z}/(q^n - 1)\mathbb{Z}.$$

Assuming that $R_2(F_q)$ is generated by five term relations and that $R_3(F_q)$ is generated by the 31 term relations in [Zic19], one can compute $H^1(\wedge^*(\mathbb{L}(F_q))_n)$ for $n = 2$ and 3. When $q > 3$, experimental evidence suggests that *integrally*, one has

$$(39) \quad |H^1(\wedge^*(\mathbb{L}(F_q))_n)| \stackrel{=2,3}{=} \frac{|H_{\mathcal{M}}^1(F_q, \mathbb{Z}(n))|}{|F_q^*|}$$

where $\stackrel{=2,3}{=}$ means equality up to small powers of 2 and 3. The equality appears to hold on the nose when $n = 2$.

4. INVERSION RELATIONS

In the following we ignore 2-torsion. It is well known (see e.g. [Lew91]) that the classical polylogarithms satisfy the *inversion relation*

$$(40) \quad \text{Li}_n(x) + (-1)^n \text{Li}_n(x^{-1}) = -\frac{(2\pi i)^n}{n!} B_n \left(\frac{\log x}{2\pi i} \right),$$

where $B_n(x) = \sum_{k=0}^n \binom{n}{k} B_{n-k} x^k$ are the Bernoulli polynomials. Thinking of a symbol $[x]_n$ as $\text{Li}_n(x)$ modulo products and powers of πi , we would thus expect $[x]_n + (-1)^n [x^{-1}]_n$ to be zero in $\mathbb{L}_n(F)$. This was proved in Example 3.5.

Goncharov [Gon01, Sec. 2.6] extended the classical inversion formula to multiple polylogarithms. For example, one has

$$(41) \quad \begin{aligned} & \text{Li}_{n_1, n_2}(x_1, x_2) + (-1)^{n_1+n_2} \text{Li}_{n_2, n_1}(x_2^{-1}, x_1^{-1}) + (-1)^{n_1} \text{Li}_{n_1}(x_1^{-1}) \text{Li}_{n_2}(x_2) \\ & + \sum_{p+q=n_1} \frac{(2\pi i)^p}{p!} (-1)^q \binom{q+n_2-1}{n_2-1} B_p \left(\frac{\log(x_1 x_2)}{2\pi i} \right) \text{Li}_{q+n_2}(x_2) \\ & + \sum_{p+q=n_2} \frac{(2\pi i)^p}{p!} (-1)^{n_1} \binom{n_1+q-1}{n_1-1} B_p \left(\frac{\log(x_1 x_2)}{2\pi i} \right) \text{Li}_{n_1+q}(x_1^{-1}) = 0, \end{aligned}$$

which suggests that we should have

$$(42) \quad \begin{aligned} & [x_1, x_2]_{n_1, n_2} + (-1)^{n_1+n_2} [x_2^{-1}, x_1^{-1}]_{n_2, n_1} \\ & + (-1)^{n_1} \binom{n_1+n_2-1}{n_2-1} [x_2]_{n_1+n_2} + (-1)^{n_1} \binom{n_1+n_2-1}{n_1-1} [x_1^{-1}]_{n_1+n_2} = 0 \in \mathbb{L}_{n_1+n_2}(F). \end{aligned}$$

More generally, Goncharov's work suggests that $[x_d^{-1}, \dots, x_1^{-1}]_{n_d, \dots, n_1}$ should be expressible using terms where no x_i is inverted. Inspired by Goncharov's work we make the following definition.

Definition 4.1. The *inversion map* is defined inductively on power series by the formula

$$(43) \quad \text{inv}([\mathbf{x}_{1,\dots,d} | \mathbf{t}_{1,\dots,d}]) = -(-1)^d [\mathbf{x}_{1,\dots,d} | \mathbf{t}_{1,\dots,d}] + \frac{(-1)^d}{t_1} [\mathbf{x}_{2,\dots,d} | \mathbf{t}_{2,\dots,d}] \\ - \frac{(-1)^d}{t_1} [\mathbf{x}_{2,\dots,d} | \mathbf{t}_{2,\dots,d} - t_1] - \frac{1}{t_d} \text{inv}([\mathbf{x}_{1,\dots,d-1} | \mathbf{t}_{1,\dots,d-1}]) + \frac{1}{t_d} \text{inv}([\mathbf{x}_{1,\dots,d-1} | \mathbf{t}_{1,\dots,d-1} - t_d]).$$

The induction starts with the formula $\text{inv}[x|t] = [x|t] + [x]_0$.

We wish to prove the following:

Theorem 4.2. We have $[\mathbf{x}_{d,\dots,1}^{-1} | -\mathbf{t}_{d,\dots,1}] = \text{inv}[\mathbf{x}_{1,\dots,d} | \mathbf{t}_{1,\dots,d}] \in \mathbb{L}(F)$. In particular,

$$(44) \quad (-1)^d (-1)^{n_1+\dots+n_d} [x_d^{-1}, \dots, x_1^{-1}]_{n_d, \dots, n_1} = \text{inv}[x_1, \dots, x_d]_{n_1, \dots, n_d}.$$

Example 4.3. For $d = 1$, we have $(-1)^{n+1} [x^{-1}]_n = [x]_n$ for $n > 1$ and $[x^{-1}]_1 = [x]_1 + [x]_0$. These hold by Example 3.5.

Example 4.4. For $d = 2$, $\text{inv}[x_1, x_2 | t_1, t_2]$ equals

$$(45) \quad -[x_1, x_2 | t_1, t_2] + \frac{1}{t_1} [x_2 | t_2] - \frac{1}{t_1} [x_2 | t_2 - t_1] - \frac{1}{t_2} \text{inv}[x_1 | t_1] + \frac{1}{t_2} \text{inv}[x_1 | t_1 - t_2].$$

The coefficient $t_1^{n_1-1} t_2^{n_2-1}$ of the equation in Theorem 4.2, gives (42).

Recall that the proof that $[x]_n + (-1)^n [x^{-1}]_n = 0$ used that the symbol $[\infty]_n$ is well defined and equal to 0. Unfortunately, $\mathcal{L}_{n_1, \dots, n_d}(x_1, \dots, x_d)$ does not tend to 0 when some of the x_i tend to ∞ , so we do not expect to be able to simply declare $[x_1, \dots, x_d]_{n_1, \dots, n_d}$ to be 0 when some of the x_i are ∞ . Instead we take the equality in Theorem 4.2 as a *definition*.

Definition 4.5. If some of the x_i are ∞ , we *define*

$$(46) \quad [x_1, \dots, x_d]_{n_1, \dots, n_d} = (-1)^d (-1)^{n_1+\dots+n_d} \text{inv}([x_d^{-1}, \dots, x_1^{-1}]_{n_1, \dots, n_d})$$

when $n_1 + \dots + n_d > 1$.

For example we have

$$(47) \quad [x_1, \infty]_{n_1, n_2} = (-1)^{n_2} \binom{n_1 + n_2}{n_1 - 1} [x_1]_{n_1 + n_2}, \quad [\infty, x_2]_{n_1, n_2} = -(-1)^{n_1} \binom{n_1 + n_2}{n_2 - 1} [x_2]_{n_1 + n_2}$$

Remark 4.6. In weight 1, symbols $[\infty]_1$ and $[\infty]_0$ are not defined.

Proof of Theorem 4.2. The structure of the proof is similar to that of the depth 1 inversion relation in Example 3.5. Suppose by induction that it holds in depth less than d and in weight less than n . The case of depth 1 is the relation $[x]_n + (-1)^n [x^{-1}]_n = 0$. For $X = [\mathbf{x}_{1,\dots,d} | \mathbf{t}_{1,\dots,d}]$ let

$$(48) \quad X^{-1} = [\mathbf{x}_{d,\dots,1}^{-1} | -\mathbf{t}_{d,\dots,1}].$$

We must prove that $X^{-1} = \text{inv}(X)$. We first prove by induction that $\delta(X^{-1})_n = \delta(\text{inv } X)_n$, which implies that $X^{-1} - \text{inv}(X)$ is constant. Setting x_1, \dots, x_d equal to zero then gives the result by Definition 4.5. In order to prove that $\delta(X^{-1})_n = \delta(\text{inv}(X))_n$, we consider all possible terms involved. It will be convenient to think of the x_i appearing in each term as formal variables, and not elements in F . It thus makes sense to talk about *inverted terms* (those that involve inverses x_i^{-1}) and *regular terms* (those that don't involve inverses). Any identity in the free abelian group on terms (or their wedge products) gives rise to an identity in $\mathbb{L}^{\text{symb}}(F)$ (or $\wedge^2(\mathbb{L}^{\text{symb}}(F))$) by

assigning values to the x_i . We shall consider three distinct operations on terms which we all think of as inversions:

$$(49) \quad \text{inv}, \quad X \mapsto X^{-1}, \quad \iota,$$

where inv is defined by (43), $X \mapsto X^{-1}$ by (48), and ι by fixing regular terms and replacing an inverted term X^{-1} by $\text{inv}(X)$. Letting

$$(50) \quad \begin{aligned} A &= [\mathbf{x}_{2,\dots,d} | \mathbf{t}_{2,\dots,d}] & B &= [\mathbf{x}_{2,\dots,d} | \mathbf{t}_{2,\dots,d} - t_1], \\ C &= [\mathbf{x}_{1,\dots,d-1} | \mathbf{t}_{1,\dots,d-1}], & D &= [\mathbf{x}_{1,\dots,d-1} | \mathbf{t}_{1,\dots,d-1} - t_d]. \end{aligned}$$

we claim that the following holds in $\mathbb{L}^{\text{symb}}(F)$:

$$(51) \quad \iota(\delta(X^{-1}) + (-1)^d \delta(X)) = \iota \delta \left(\frac{(-1)^d}{t_1} A - \frac{(-1)^d}{t_1} B - \frac{1}{t_d} C^{-1} + \frac{1}{t_d} D^{-1} \right).$$

Since the cobracket of X consists entirely of terms of lower depth and weight, where $Y^{-1} = \text{inv}(Y)$ by induction, it follows that one has

$$(52) \quad \begin{aligned} \delta(X^{-1})_n &= \delta \left(-(-1)^d X + \frac{(-1)^d}{t_1} A - \frac{(-1)^d}{t_1} B - \frac{1}{t_d} C^{-1} + \frac{1}{t_d} D^{-1} \right)_n = \\ &= \delta \left(-(-1)^d X + \frac{(-1)^d}{t_1} A - \frac{(-1)^d}{t_1} B - \frac{1}{t_d} \text{inv}(C) + \frac{1}{t_d} \text{inv}(D) \right)_n = \delta \text{inv}(X)_n \end{aligned}$$

in $\mathbb{L}_n(F)$. It remains to prove (51). Using the shorthands

$$(53) \quad \begin{aligned} Y_{r,s} &= [\mathbf{x}_{r,\dots,s} | \mathbf{t}_{r,\dots,s}], & Y_{r,s;u} &= [\mathbf{x}_{r,\dots,s} | \mathbf{t}_{r,\dots,s} - t_u] \\ Y_{r,s}^{p \rightarrow q} &= [\mathbf{x}_{r,\dots,p \rightarrow q,\dots,s} | \mathbf{t}_{r,\dots,p,q+1,\dots,s}], & Y_{r,s;u}^{p \rightarrow q} &= [\mathbf{x}_{r,\dots,p \rightarrow q,\dots,s} | \mathbf{t}_{r,\dots,p,q+1,\dots,s} - t_u] \\ Z_{r,s}^{p \rightarrow q} &= [\mathbf{x}_{r,\dots,p \rightarrow q,\dots,s} | \mathbf{t}_{r,\dots,p-1,q,\dots,s}], & Z_{r,s;u}^{p \rightarrow q} &= [\mathbf{x}_{r,\dots,p \rightarrow q,\dots,s} | \mathbf{t}_{r,\dots,p-1,q,\dots,s} - t_u] \end{aligned}$$

we have

$$(54) \quad \begin{aligned} \delta(X^{-1}) + (-1)^d \delta(X) &= \left(X + (-1)^d X^{-1} \right) \wedge \sum_{p=1}^d t_p [x_p]_0 + \\ &= \sum_{2 \leq p \leq d} (-1)^p \left(Y_{p,d} + (-1)^{d-p+1} Y_{p,d}^{-1} \right) \wedge Y_{1,p-1}^{-1} + \\ &= \sum_{2 \leq p \leq d} Y_{p,d} \wedge \left(Y_{1,p-1} + (-1)^{p-1} Y_{1,p-1}^{-1} \right) + \\ &= \sum_{1 \leq p < q \leq d} \left(Y_{1,d}^{p \rightarrow q} + (-1)^{d-q+p} (Y_{1,d}^{p \rightarrow q})^{-1} \right) \wedge Y_{p+1,q;p} + \\ &= \sum_{1 \leq p < q \leq d} (-1)^{q-p} \left(Z_{1,d}^{p \rightarrow q} + (-1)^{d-q+p} (Z_{1,d}^{p \rightarrow q})^{-1} \right) \wedge Y_{p,q-1;q}^{-1}. \end{aligned}$$

For any regular term W we can define A_W , B_W , C_W , and D_W as in (50). If W has depth d_W the definitions of ι and inv imply that

$$(55) \quad \iota(W^{-1} + (-1)^{d_W} W) = \iota \left(\frac{(-1)^{d_W}}{t_1} A_W - \frac{(-1)^{d_W}}{t_1} B_W - \frac{1}{t_{d_W}} C_W^{-1} + \frac{1}{t_{d_W}} D_W \right).$$

Plugging this into 54 it is now straightforward to match up the terms with those on the righthand side of (51). This concludes the proof. \square

Remark 4.7. Even if we don't allow any symbols with $x_i = \infty$ we still expect Theorem 4.2 to hold, but only up to torsion. For example, one can show that $6([x]_2 + [x^{-1}]_2) = 0 \in \mathbb{L}_2(F)$.

4.1. A coalgebra without inverted terms. As an aside, we now show that if we change δ by replacing each term $[x_{q-1}^{-1}, \dots, x_p^{-1}|t_q - \mathbf{t}_{p, \dots, q-1}]$ by $\text{inv}[\mathbf{x}_{p, \dots, q-1}|\mathbf{t}_{p, \dots, q-1} - t_q]$, we obtain another cobracket δ' on $\mathbb{L}^{\text{symb}}(F)$. Moreover, the groups $R_n(F)$ defined using δ or δ' are the same. All that remains is to show that $\delta'^2 = 0$. As in the proof of Theorem 4.2 the argument is purely symbolic, and makes use of the operation ι .

Lemma 4.8. For a regular term X we have

$$(56) \quad \iota \delta(X) = \delta'(X), \quad \iota \delta(X^{-1}) = \delta'(\text{inv}(X)).$$

Proof. The first equality follows immediately from the definition of δ' . The second holds in depth 1, and the general case follows by induction from (51). \square

Theorem 4.9. δ' is a cobracket on $\mathbb{L}^{\text{symb}}(F)$, i.e. $\delta'^2 = 0$.

Proof. Let's write $\delta(X)$ in the form $\sum_i X_1^{(i)} \wedge X_2^{(i)} + \sum_j X_3^{(j)} \wedge X_4^{(j)-1}$. By Lemma 4.8 we have

$$(57) \quad \begin{aligned} 0 &= \iota \delta^2(X) = \iota \delta \left(\sum_i X_1^{(i)} \wedge X_2^{(i)} + \sum_j X_3^{(j)} \wedge X_4^{(j)-1} \right) \\ &= \sum_i \iota \delta X_1^{(i)} \wedge X_2^{(i)} - X_1^{(i)} \wedge \iota \delta X_2^{(i)} + \sum_j \iota \delta X_3^{(j)} \wedge \text{inv} X_4^{(j)} - X_3^{(j)} \wedge \iota \delta X_4^{(j)-1} \\ &= \sum_i \delta' X_1^{(i)} \wedge X_2^{(i)} - X_1^{(i)} \wedge \delta' X_2^{(i)} + \sum_j \delta' X_3^{(j)} \wedge \text{inv} X_4^{(j)} - X_3^{(j)} \wedge \delta' \text{inv} X_4^{(j)} \\ &= \delta' \left(\sum_i X_1^{(i)} \wedge X_2^{(i)} + \sum_j X_3^{(j)} \wedge \text{inv} X_4^{(j)} \right) = \delta'^2(X). \end{aligned}$$

This proves the result. \square

5. SHUFFLE RELATIONS

Since the multiple polylogarithms are iterated integrals they satisfy shuffle relations (see e.g. [Gon01, Sec. 2.5]). For example, one has

$$(58) \quad \text{Li}_{n_1}(x_1) \text{Li}_{n_2}(x_2) = \text{Li}_{n_1, n_2}(x_1, x_2) + \text{Li}_{n_2, n_1}(x_2, x_1) + \text{Li}_{n_1+n_2}(x_1 x_2).$$

This motivates us to define *shuffle products*

$$(59) \quad [x_1]_{n_1} [x_2]_{n_2} = [x_1, x_2]_{n_1, n_2} + [x_2, x_1]_{n_2, n_1} + [x_1 x_2]_{n_1+n_2} \in \mathbb{L}_{n_1+n_2}(F).$$

In terms of power series this is equivalent to

$$(60) \quad [x_1|t_1][x_2|t_2] = [x_1, x_2|t_1, t_2] + [x_2, x_1|t_2|t_1] + \frac{1}{t_1 - t_2} ([x_1 x_2|t_1] - [x_1 x_2|t_2]).$$

Similarly, we define a shuffle product

$$(61) \quad \begin{aligned} [x_1, x_2|t_1, t_2][x_3|t_3] &= [x_1, x_2, x_3|t_1, t_2, t_3] + [x_1, x_3, x_2|t_1, t_3, t_2] + [x_3, x_1, x_2|t_3, t_1, t_2] \\ &+ \frac{1}{t_1 - t_3} ([x_1 x_3, x_2|t_1, t_2] - [x_1 x_3, x_2|t_3, t_2]) + \frac{1}{t_2 - t_3} ([x_1, x_2 x_3|t_1, t_2] - [x_1, x_2 x_3|t_1, t_3]) \end{aligned}$$

corresponding to the shuffle relations for $\text{Li}_{n_1, n_2}(x_1, x_2) \text{Li}_{n_3}(x_3)$.

Theorem 5.1. *The shuffle products $[x_1|t_1][x_2|t_2]$ and $[x_1, x_2|t_1, t_2][x_3, t_3]$ are zero in $\mathbb{L}(F)$.*

Proof. A simple computation shows that

$$(62) \quad \delta([x_1|t_1][x_2|t_2]) = ([x_1|t_1][x_2|t_2]) \wedge (t_1[x_1]_0 + t_2[x_2]_0) \in \wedge^2(\mathbb{L}(F)).$$

Assuming by induction that $([x_1|t_1][x_2|t_2])_{n-1} = 0$ it follows that $([x_1|t_1][x_2|t_2])_n$ is constant. Setting $x_1 = x_2 = 0$ shows that the constant is 0. Another straightforward computation shows that

$$(63) \quad \delta([x_1, x_2|t_1, t_2][x_3|t_3]) = ([x_1, x_2|t_1, t_2][x_3|t_3]) \wedge \sum_{i=1}^3 t_i[x_i]_0 - [x_1|t_1] \wedge ([x_2|t_2][x_3|t_3]) \\ + [x_2|t_2] \wedge ([x_1|t_1][x_3|t_3]) + ([x_1x_2|t_1][x_3|t_3]) \wedge [x_2|t_2 - t_1] - ([x_1x_2|t_2][x_3|t_3]) \wedge ([x_1|t_1 - t_2] + [x_1]_0) \\ + [x_1x_2x_3|t_1] \wedge ([x_2|t_2 - t_1][x_3|t_3 - t_1]) - [x_1x_2x_3|t_2] \wedge ([x_1|t_1 - t_2][x_3|t_3 - t_2]),$$

so the same argument as above shows that $[x_1, x_2|t_1, t_2][x_3, t_3] = 0$ as well. \square

Remark 5.2. We can similarly define shuffle products in arbitrary depth. We conjecture that they are all zero in $\mathbb{L}(F)$. Proving this would require showing that the cobracket of a shuffle relation is given in terms of shuffle relations in lower weight and depth. This appears to be the case.

Remark 5.3. It seems worth noting that if one replaces δ by δ' , (62) and (63) hold not just in $\wedge^2(\mathbb{L}(F))$, but also $\wedge^2(\mathbb{L}^{\text{symb}}(F))$ (and not just modulo 2-torsion).

6. THE GONCHAROV–RUDENKO LIE COALGEBRA

Goncharov and Rudenko [GR18] also considered the problem of explicitly constructing the motivic Lie-coalgebra. They give an explicit construction in weight 4 and less. We denote their coalgebra $\mathbb{L}_{\leq 4}^{GR}(F)$. They define $\mathbb{L}_1^{GR}(F) = F^*$, $\mathbb{L}_2^{GR}(F) = \mathcal{B}_2(F)$, and define (for $n = 3, 4$) $\mathbb{L}_n^{GR}(F)$ to be the group generated by symbols $\{x\}_n$ and $\{x, y\}_{n-1,1}$ subject to certain relations, most notably a relation Q_n . The relationship with their work and ours is given in Theorem 6.1 below. Its proof is straightforward, but long, so we omit it. The main point is to show that Q_n maps to an element in $R_n(F)$. We stress that their symbol $\{x, y\}_{n-1,1}$ does not correspond to our symbol $[x, y]_{n-1,1}$. We find our symbol more natural since it directly relates to the multiple polylogarithm (their symbol is related to motivic correlators).

Theorem 6.1. *There is a surjective map $\mathbb{L}_{\leq 4}^{GR}(F)_{\mathbb{Q}} \rightarrow \mathbb{L}_{\leq 4}(F)_{\mathbb{Q}}$ taking $\{x\}_n$ to $[x]_n$, $\{x, y\}_{2,1}$ to $-[x/y, y]_{2,1} - [x]_3 - [y]_3$ and $\{x, y\}_{3,1}$ to $-[x/y, y]_{3,1} - [x]_4 + [y]_4$. The map preserves the cobracket.*

Remark 6.2. We suspect that the map is an isomorphism, but we do not know if all of our relations can be expressed in terms of the Q_n relations.

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