

HOLOMORPHIC POLYLOGARITHMS AND BLOCH COMPLEXES

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ABSTRACT. For an integer $n \geq 2$ we define a polylogarithm $\widehat{\mathcal{L}}_n$, which is a holomorphic function on the universal abelian cover of $\mathbb{C} \setminus \{0, 1\}$ defined modulo $(2\pi i)^n / (n-1)!$. We use the formal properties of its functional relations to define groups $\widehat{\mathcal{B}}_k(\widehat{F})$ lifting Goncharov's Bloch groups $\mathcal{B}_k(F)$ of a field F , and show that they fit into a complex $\widehat{\Gamma}(F, n)$ lifting Goncharov's Bloch complex $\Gamma(F, n)$. When $F = \mathbb{C}$ we show that the imaginary part (when n is even) or real part (when n is odd) of $\widehat{\mathcal{L}}_n$ agrees with a real single valued polylogarithm \mathcal{L}_n on the group $H^1(\widehat{\Gamma}(\mathbb{C}, n))$. When $n = 2$, this group is Neumann's extended Bloch group. Goncharov's complex conjecturally computes the rational motivic cohomology of F , and one may speculate whether the extended complex computes the integral motivic cohomology. Finally, we use $\widehat{\mathcal{L}}_3$ to construct a lift of Goncharov's map $H_5(\mathrm{SL}(3, \mathbb{C})) \rightarrow \mathbb{R}$ to a complex valued map whose real part agrees with that of Goncharov. The lift makes use of the cluster ensemble structure on the Grassmannian $\mathrm{Gr}(3, 6)$.

1. INTRODUCTION

For a natural number n , the polylogarithm of weight n is defined by the power series

$$(1.1) \quad \mathrm{Li}_n(z) = \sum_{k=1}^{\infty} z^k / k^n, \quad |z| \leq 1.$$

It extends holomorphically to $\mathbb{C} \setminus (1, \infty)$, but is multivalued on \mathbb{C} with branch points at 0 and 1. There are several real single valued analogues of the polylogarithm (see [Zag91] for definitions and basic properties). We shall only consider

$$(1.2) \quad \mathcal{L}_n(z) = \mathfrak{R}_n \left(\sum_{r=0}^{n-1} \frac{2^r B_r}{r!} \mathrm{Li}_{n-r}(z) (\log |z|)^r \right),$$

where $\mathfrak{R}_n(x)$ denotes the real part of x when n is odd and the imaginary part when n is even, and $B_0 = 1$, $B_1 = -1/2$, $B_2 = 1/6$, $B_3 = 0$, $B_4 = -1/30$, etc., are the Bernoulli numbers. The functions $\mathcal{L}_n(z)$ are continuous on $\mathbb{C}P^1 = \mathbb{C} \cup \{\infty\}$, and

$$(1.3) \quad \mathcal{L}_2(z) = \mathrm{Im}(\mathrm{Li}_2(z)) + \mathrm{Im}(\log(1-z)) \log(|z|)$$

is the Bloch-Wigner dilogarithm.

Notation 1.1. For a field F , $F^* = F \setminus \{0\}$ denotes the unit group and $P_F^1 = F \cup \{\infty\}$. For an abelian group A , $A_{\mathbb{Q}}$ denotes $A \otimes \mathbb{Q}$. All tensor products and exterior powers are over \mathbb{Z} unless otherwise specified. For a set X , $\mathbb{Z}[X]$ denotes the free abelian group generated by X , and for $x \in X$ the corresponding generator in $\mathbb{Z}[X]$ is denoted $[x]$.

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For a field F , Goncharov [Gon95] has defined complexes $\Gamma(F, n)$ of the form

$$(1.4) \quad \mathcal{B}_n(F) \xrightarrow{\delta} \cdots \xrightarrow{\delta} \mathcal{B}_{n-k}(F) \otimes \wedge^k(F^*) \xrightarrow{\delta} \cdots \xrightarrow{\delta} \mathcal{B}_2(F) \otimes \wedge^{n-2}(F^*) \xrightarrow{\delta} \wedge^n(F^*).$$

Each group $\mathcal{B}_k(F)$ is the quotient of $\mathbb{Z}[P_F^1]$ by a subgroup $R_k(F)$, which is defined inductively, and may be thought of as formal functional relations for \mathcal{L}_k . For example, for any $x, y \in F$ we have an element (where $\frac{0}{0} = 1$, $\frac{1}{0} = \infty$, etc.)

$$(1.5) \quad [x] - [y] + \left[\frac{y}{x}\right] - \left[\frac{1-x^{-1}}{1-y^{-1}}\right] + \left[\frac{1-x}{1-y}\right] \in R_2(F)$$

which corresponds to the well known functional relation

$$(1.6) \quad \mathcal{L}_2(x) - \mathcal{L}_2(y) + \mathcal{L}_2\left(\frac{y}{x}\right) - \mathcal{L}_2\left(\frac{1-x^{-1}}{1-y^{-1}}\right) + \mathcal{L}_2\left(\frac{1-x}{1-y}\right) = 0, \quad x, y \in \mathbb{C}.$$

Conjecture 1.2 (Goncharov [Gon95, Conj. 1.20]). For $n = 2$ and $n = 3$, we have

- a) $R_2(F)$ is generated by the five term relations (1.5).
- b) $R_3(F)$ is generated by the elementary relations $[x] - [x^{-1}]$, and $[x] + [1-x] + [1-x^{-1}] - [1]$ together with a 3-variable relation of the form $R_3(x, y, z) - 3[1]$, where $R_3(x, y, z)$ is an explicit element with 22 terms (see also [Zag91]).

Remark 1.3. For $n > 3$ few elements in $R_n(F)$ are known other than $[x] + (-1)^n[x^{-1}]$ which is in $R_n(F)$ for any n . Gangl [Gan16] has constructed a 931 term relation in $R_4(F)$; see also [GR18] for an approach to describing $R_4(F)$ via cluster algebras. A discussion of Conjecture 1.2a) is given in [dJ20].

Conjecture 1.4 (Goncharov [Gon94, Conj. 2.1]). There are rational isomorphisms

$$(1.7) \quad H^i(\Gamma(F, n))_{\mathbb{Q}} \cong K_{2n-i}^{(n)}(F)_{\mathbb{Q}}, \quad i = 1, \dots, n$$

where $K_q^{(p)}(F) = \text{gr}_{\gamma}^p K_q(F)$ are the associated graded groups for the γ -filtration on $K_q(F)$.

Goncharov further speculates (see [Gon05b]) that when $F = \mathbb{C}$, the map \mathcal{L}_n (defined on $\mathcal{B}_n(\mathbb{C})$ by linear extension) agrees with the Borel regulator map b_n , i.e. that there is a commutative diagram

$$(1.8) \quad \begin{array}{ccc} K_{2n-1}^{(n)}(\mathbb{C})_{\mathbb{Q}} & \xrightarrow{\cong} & H^1(\Gamma(\mathbb{C}, n))_{\mathbb{Q}} \\ & \searrow b_n & \swarrow \mathcal{L}_n \\ & \mathbb{R} & \end{array}$$

Remark 1.5. The groups $K_q^{(p)}(F)$ are rationally isomorphic to Blochs higher Chow groups $CH^p(F, q)$. In fact, one has $K_q^{(p)}(F)_{\mathbb{Q}} \cong CH^p(F, q)_{\mathbb{Q}}$ [Lev97].

1.1. Motivic cohomology. For a smooth scheme X over a field F , Voevodsky has defined motivic cohomology groups $H_{\mathcal{M}}^i(X, \mathbb{Z}(n))$ satisfying (see [Voe02, MVW06])

$$(1.9) \quad H_{\mathcal{M}}^i(X, \mathbb{Z}(n)) \cong CH^n(X, 2n-i), \quad H^n(F, \mathbb{Z}(n)) = K_n^M(F),$$

where $K_n^M(F)$ is Milnor K -theory. By Remark 1.5 we reformulate Conjecture 1.4:

Conjecture 1.6.

$$(1.10) \quad H^i(\Gamma(F, n))_{\mathbb{Q}} \cong H_{\mathcal{M}}^i(F, \mathbb{Z}(n))_{\mathbb{Q}}.$$

Example 1.7. When $n = 2$ the complex $\Gamma(F, 2)$ takes the form

$$(1.11) \quad \delta: \mathcal{B}_2(F) \rightarrow \wedge^2(F^*), \quad [x] \mapsto x \wedge (1 - x).$$

Assuming Conjecture 1.2a), the kernel of δ is (up to 6 torsion) the classical Bloch group $\mathcal{B}(F)$, and the cokernel is $K_2^M(F)$ by Matsumoto's theorem. We also have (see [Sus87])

$$(1.12) \quad H_{\mathcal{M}}^1(F, \mathbb{Z}(2)) \cong CH^2(F, 3) \cong K_3^{\text{ind}}(F), \quad H_{\mathcal{M}}^2(F, \mathbb{Z}(2)) \cong K_2^M(F).$$

Since $K_3^{\text{ind}}(F)$ is an extension of $\mathcal{B}(F)$ by a non-trivial torsion group [Sus90], it follows that Conjecture 1.2a) implies Conjecture 1.6 when $n = 2$, and that Conjecture 1.6 is not true *integrally*.

For a smooth scheme X over F , Bloch [Blo86] defined cycle maps from $CH^n(X, 2n - i)$ to the Deligne cohomology group $H_{\mathcal{D}}^i(X, \mathbb{Z}(n))$ (see e.g. [EV88] for definition and basic properties of Deligne cohomology). Since $H_{\mathcal{D}}^1(\text{Spec}(\mathbb{C}), \mathbb{Z}(n)) = \mathbb{C}/(2\pi i)^n \mathbb{Z}$ it follows that there is a map

$$(1.13) \quad B_n: H_{\mathcal{M}}^1(\mathbb{C}; \mathbb{Z}(n)) \rightarrow \mathbb{C}/(2\pi i)^n \mathbb{Z}.$$

Remark 1.8. In addition to the cycle maps B_n , one also has maps $K_{2n-1}^{(n)}(\mathbb{C}) \rightarrow \mathbb{C}/(2\pi i)^n \mathbb{Z}$ defined as the composition of the Hurevicz map $K_{2n-1}(\mathbb{C}) \rightarrow H_n(\text{GL}(\mathbb{C}))$ with the Cheeger-Chern-Simons class $\hat{c}_n: H_n(\text{GL}(\mathbb{C})) \rightarrow \mathbb{C}/(2\pi i)^n \mathbb{Z}$ (see e.g. [CS85]). The two maps probably correspond, but the author is not aware of any proof of this.

1.2. Motivating goals. A motivating goal for our work is to obtain a variant of Conjecture 1.6, which holds integrally and a variant of (1.8) involving B_n instead of b_n . In other words, we want a complex $\widehat{\Gamma}(F, n)$ such that

- (1) $H^i(\widehat{\Gamma}(F, n))$ is *integrally* isomorphic to the motivic cohomology group $H_{\mathcal{M}}^i(F; \mathbb{Z}(n))$.
- (2) When $F = \mathbb{C}$, the map B_n is given by an explicit polylogarithm on $H^1(\widehat{\Gamma}(\mathbb{C}, n))$.

1.3. Neumann's extended Bloch group. When $n = 2$ and $F = \mathbb{C}$, one may interpret work of Neumann [Neu04] (after modifications [GZ07, Zic15]; see Remark 1.9 for a discussion) as an achievement of our goals. Neumann [Neu04] considered the universal abelian cover (also considered by Zagier [Zag07])

$$(1.14) \quad r: \widehat{\mathbb{C}} = \{(u, v) \in \mathbb{C}^2 \mid e^u + e^v = 1\} \rightarrow \mathbb{C} \setminus \{0, 1\}, \quad (u, v) \mapsto e^u$$

of $\mathbb{C} \setminus \{0, 1\}$ and defined an explicit dilogarithm function $R: \widehat{\mathbb{C}} \rightarrow \mathbb{C}/4\pi^2 \mathbb{Z}$ satisfying five term relations lifting the relations (1.5) (when $x \neq y \in \mathbb{C} \setminus \{0, 1\}$). Letting $\widehat{\mathcal{P}}(\mathbb{C})$ be the free abelian group on $\widehat{\mathbb{C}}$ modulo the lifted five term relations, Neumann defined a map (c.f. (1.11))

$$(1.15) \quad \widehat{\nu}: \widehat{\mathcal{P}}(\mathbb{C}) \rightarrow \wedge^2(\mathbb{C}), \quad [(u, v)] \mapsto u \wedge v,$$

and showed (see [GZ07]) that there are isomorphisms

$$(1.16) \quad \text{Ker}(\widehat{\nu}) \cong H_3(\text{SL}(2, \mathbb{C})), \quad \text{Coker}(\widehat{\nu}) \cong K_2^M(\mathbb{C}),$$

and that the map $\text{Ker}(\widehat{\nu}) \rightarrow \mathbb{C}/4\pi^2 \mathbb{Z}$ induced by R agrees with the second Cheeger-Chern-Simons class $\hat{c}_2: H_3(\text{SL}(2, \mathbb{C})) \rightarrow \mathbb{C}/4\pi^2 \mathbb{Z}$. Since $H_3(\text{SL}(2, \mathbb{C}))$ is isomorphic to $K_3^{\text{ind}}(\mathbb{C})$ [Sah89] it follows from (1.12) that (1.15) may be viewed as a model for $\widehat{\Gamma}(\mathbb{C}, 2)$. The group $\text{Ker}(\widehat{\nu})$ is called the *extended Bloch group* and is denoted $\widehat{\mathcal{B}}(\mathbb{C})$.

1.3.1. *Arbitrary fields.* Although Neumann's work relies on analytic continuation it was generalized to arbitrary fields by Zickert [Zic15]. Given a torsion free \mathbb{Z} -extension E of F^* , Zickert defined a group $\widehat{\mathcal{P}}_E(F)$ and a map

$$(1.17) \quad \widehat{\nu}: \widehat{\mathcal{P}}_E(F) \rightarrow \wedge^2(E)$$

with cokernel $K_2^M(F)$. When $F = \mathbb{C}$ and $E = \mathbb{C}$ is the extension of \mathbb{C}^* given by the exponential map, (1.17) agrees with Neumann's complex (1.15). The corresponding extended Bloch group $\widehat{\mathcal{B}}_E(F) = \text{Ker}(\widehat{\nu})$ only depends on the class of E in $\text{Ext}(F^*, \mathbb{Z})$, and if F^* is free modulo torsion and has finitely many roots of unity, $\widehat{\mathcal{B}}_E(F)$ is independent of E up to natural isomorphism. If so, we suppress E from the notation. If, in addition, F admits an embedding in \mathbb{C} , the extended Bloch group $\widehat{\mathcal{B}}(F)$ is isomorphic to $K_3^{\text{ind}}(F)$. In particular, (1.17) is a model for $\widehat{\Gamma}(F, 2)$ whenever F is a finite extension of \mathbb{Q} .

1.3.2. *Variants.* Neumann defined two variants of his extended Bloch group. The other variant is defined using the disconnected cover

$$(1.18) \quad \widehat{\mathbb{C}}_{\pm} = \left\{ (u, v) \in \mathbb{C}^2 \mid \epsilon_1 e^u + \epsilon_2 e^v = 1 \text{ for some } \epsilon_1, \epsilon_2 \in \{-1, 1\} \right\}.$$

On this cover, the lifted five term relations are only defined modulo $\pi^2\mathbb{Z}$, and the resulting extended Bloch group $\widehat{\mathcal{B}}(\mathbb{C})_{\pm}$ is the quotient of $\widehat{\mathcal{B}}(\mathbb{C})$ by a cyclic subgroup of order 4. The main advantage of $\widehat{\mathcal{B}}(\mathbb{C})_{\pm}$ is that elements seem to arise more naturally from other contexts. For example, a cusped hyperbolic 3-manifold M with an ideal triangulation naturally and explicitly determines an element in $\widehat{\mathcal{B}}(\mathbb{C})_{\pm}$. A variant for arbitrary fields was defined by Zickert [Zic15].

Remark 1.9. We stress that our notation differs from that of Neumann [Neu04]. For example, Neumann used $\widehat{\mathcal{B}}(\mathbb{C})$ to denote the variant using $\widehat{\mathbb{C}}_{\pm}$. Also, Neumann's original R was only defined modulo π^2 and defined on $H_3(\text{PSL}(2, \mathbb{C}))$ instead of $H_3(\text{SL}(2, \mathbb{C}))$. Our notation largely follows Zickert [Zic15].

1.4. **Summary of results.** In Section 2 we define a holomorphic function

$$(1.19) \quad \widehat{\mathcal{L}}_n: \widehat{\mathbb{C}} \rightarrow \mathbb{C} / \frac{(2\pi i)^n}{(n-1)!} \mathbb{Z}$$

and state its fundamental properties. The differential of $\widehat{\mathcal{L}}_n$ is the holomorphic 1-form

$$(1.20) \quad d\widehat{\mathcal{L}}_n = (-1)^n \frac{n-1}{n!} u^{n-2} (udv - vdu) \in \Omega^1(\widehat{\mathbb{C}}).$$

The $\widehat{\mathcal{L}}_n$ are also defined on the disconnected cover $\widehat{\mathbb{C}}_{\pm}$, where the ambiguity of definition is reduced by a power of 2 depending on n (see Theorem 2.2). In Section 3 we show that certain functional relations for \mathcal{L}_n give rise to analogous functional relations for $\widehat{\mathcal{L}}_n$, the idea being that the vanishing of the symbol map is equivalent to the vanishing of a certain symbolic 1-form corresponding to $d\widehat{\mathcal{L}}_n$. The $\widehat{\mathcal{L}}_n$ relations are sometimes more naturally defined on $\widehat{\mathbb{C}}_{\pm}$.

In general, the relations depend on choices of log branches, but it turns out that there is a completely combinatorial (inductive) description of relations where the value of $\widehat{\mathcal{L}}_n$ is independent of these choices. This allows us to define relations over arbitrary fields. As in Zickert [Zic15] this is done using a choice of torsion free \mathbb{Z} -extension $\pi: E \rightarrow F^*$ (if $F = \mathbb{C}$, we choose the exponential extension). Given this choice, we have a natural algebraic analogue

$$(1.21) \quad \widehat{F} = \left\{ (u, v) \in E \times E \mid \pi(u) + \pi(v) = 1 \right\}$$

of Neumann's cover $\widehat{\mathbb{C}}$. In Section 4 we define subgroups $\widetilde{R}_k(F)$ of $\mathbb{Z}[\widehat{F}]$ such that the groups $\widehat{\mathcal{B}}_k(\widehat{F}) = \mathbb{Z}[\widehat{F}]/\widetilde{R}_k(F)$ fit in a chain complex $\widehat{\Gamma}(F, n)$ of the form

$$(1.22) \quad \widehat{\mathcal{B}}_n(\widehat{F}) \xrightarrow{\delta} \dots \xrightarrow{\delta} \widehat{\mathcal{B}}_{n-k}(\widehat{F}) \otimes \wedge^k(E) \xrightarrow{\delta} \dots \xrightarrow{\delta} \widehat{\mathcal{B}}_2(\widehat{F}) \otimes \wedge^{n-2}(E) \xrightarrow{\delta} \wedge^n(E).$$

The natural map $r: \widehat{F} \rightarrow F \setminus \{0, 1\}$ taking (u, v) to $\pi(u)$ induces a chain map $\widehat{\Gamma}(F, n) \rightarrow \Gamma(F, n)$. The fact that $\widetilde{R}_n(F)$ consists of formal functional relations for $\widehat{\mathcal{L}}_n$ is proved in Section 5, and in Section 6 we show that $\mathfrak{R}_n \circ \widehat{\mathcal{L}}_n = \mathcal{L}_n \circ r$ on $H^1(\widehat{\Gamma}(\mathbb{C}, n))$. We stress that this equality only holds in cohomology and not on $\widehat{\mathbb{C}}$ (see Theorem 2.10). This is because $\widehat{\mathcal{L}}_n$ is holomorphic, but \mathcal{L}_n is not the real or imaginary part of a holomorphic function. Assuming that $\widetilde{R}_2(F)$ is generated by lifted five term relations, $\widehat{\Gamma}(F, 2)$ is Zickert's complex (1.17), so under this assumption $\widehat{\Gamma}(F, 2)$ satisfies our motivational goal. We speculate that this holds more generally, i.e. that $\widehat{\Gamma}(F, n)$ computes the motivic cohomology groups $H_{\mathcal{M}}^i(F, \mathbb{Z}(n))$ and that when $F = \mathbb{C}$ the cycle map (1.13) agrees with $(n-1)!\widehat{\mathcal{L}}_n$ under the isomorphism $H^1(\widehat{\Gamma}(\mathbb{C}, n)) \cong H_{\mathcal{M}}^1(\mathbb{C}, \mathbb{Z}(n))$.

In Section 7 we lift Goncharov's map $H_5(\mathrm{SL}(3, \mathbb{C})) \rightarrow \mathbb{R}$ to a complex valued map. Goncharov's map is obtained as a composition

$$(1.23) \quad H_5(\mathrm{SL}(3, \mathbb{C})) \rightarrow H^1(\Gamma(\mathbb{C}, 3)_{\mathbb{Q}}) \xrightarrow{\mathcal{L}_3} \mathbb{R},$$

and our map is defined similarly using $H^1(\widehat{\Gamma}(\mathbb{C}, 3))$ and $\widehat{\mathcal{L}}_3$ and is thus defined modulo $\frac{(2\pi i)^3}{2}\mathbb{Z}$. Our map is expressed in terms of \mathcal{X} -coordinates on the affine Grassmannian $\widetilde{\mathrm{Gr}}(3, 6)$, and we speculate that twice our map equals the third Cheeger-Chern-Simons class \widehat{c}_3 [CS85]. This would be a natural generalization of the fact that $\widehat{\mathcal{L}}_2$ on the extended Bloch group equals \widehat{c}_2 .

Remark 1.10. The exterior algebra $\wedge^*(A)$ of an abelian group A is regarded as the quotient of the tensor algebra by the relations $a \otimes b + b \otimes a$. In Goncharov's definition of $\Gamma(F, n)$ he additionally assumes that $x \wedge x = x \wedge (-x) = 0 \in \wedge^2(F^*)$. In the definition of $\widehat{\Gamma}(F, n)$ we assume that $a \wedge a = 0$ when $n > 2$ (to ensure that $\delta^2 = 0$). Since the variants differ only by 2-torsion we shall for notational simplicity denote them all by $\wedge^*(A)$.

Remark 1.11. We do not know the kernel and cokernel of the map $H^i(\widehat{\Gamma}(F, n)) \rightarrow H^i(\Gamma(F, n))$, but we suspect that the cokernel is trivial and that the kernel is torsion. Assuming that $\widetilde{R}_2(F)$ is generated by lifted five term relations this follows from [Zic15] when $n = 2$ and F is a finite extension of \mathbb{Q} . We also do not know how the choice of extension E affects the groups $\widehat{\mathcal{B}}_n(\widehat{F})$, but we expect that the results of Zickert for $n = 2$ (see Section 1.3.1) hold for $n > 2$ also.

Remark 1.12. We also consider a variant $\widehat{\mathcal{B}}_n(\widehat{F})_{\pm}$ defined using an algebraic analogue \widehat{F}_{\pm} of $\widehat{\mathbb{C}}_{\pm}$. The main difference is that elements are easier to produce. For example, one has the element $\alpha = [(u, v)] + (-1)^n[(-u, v - u)]$ in $\mathbb{Z}[\widehat{F}_{\pm}]$, which may be considered a lift of $\beta = [x] + (-1)^n[x^{-1}]$. The fact that $\mathcal{L}_n(\beta) = 0$ when $F = \mathbb{C}$, whereas the order of $\widehat{\mathcal{L}}_n(\alpha)$ is related to the torsion in $H_{\mathcal{M}}^1(\mathbb{Z}, \mathbb{Z}(n))$ (Corollary 2.6) provides additional support for the speculative relationship to motivic cohomology.

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2. DEFINITION AND BASIC PROPERTIES OF $\widehat{\mathcal{L}}_n$

Fix an integer $n \geq 2$. Recall the space $\widehat{\mathbb{C}}_{\pm}$ defined in (1.18). It has four components, which we denote $\widehat{\mathbb{C}}_{++}$, $\widehat{\mathbb{C}}_{--}$, $\widehat{\mathbb{C}}_{+-}$ and $\widehat{\mathbb{C}}_{-+}$ corresponding to the signs of ϵ_1 and ϵ_2 . Note that $\widehat{\mathbb{C}}_{++}$ is the space $\widehat{\mathbb{C}}$ defined in (1.14). There is a holomorphic map

$$(2.1) \quad r: \widehat{\mathbb{C}}_{\pm} \rightarrow \mathbb{C} \setminus \{0, 1\}, \quad (u, v) \mapsto \epsilon_1 e^u \quad \text{if } (u, v) \in \widehat{\mathbb{C}}_{\epsilon_1, \epsilon_2},$$

which restricts to a $\mathbb{Z} \times \mathbb{Z}$ cover on each component. On $\widehat{\mathbb{C}}_{\pm}$ we introduce the holomorphic 1-form

$$(2.2) \quad \omega_n = (-1)^n \frac{n-1}{n!} u^{n-2} (udv - vdu) \in \Omega^1(\widehat{\mathbb{C}}_{\pm}),$$

which is closed since $\widehat{\mathbb{C}}_{\pm}$ is complex 1-dimensional. Let ν_2 denote the 2-adic valuation and let

$$(2.3) \quad \kappa_n = \begin{cases} 2^{2-n} & \text{if } n \text{ is even,} \\ 2^{3+\nu_2(n-1)-n} & \text{if } n \text{ is odd.} \end{cases}$$

Theorem 2.1 (Proof in Section 2.5.1). *The form w_n has periods in $\frac{(2\pi i)^n}{(n-1)!} \mathbb{Z}$ on $\widehat{\mathbb{C}}_{++}$ and $\widehat{\mathbb{C}}_{+-}$ and periods in $\kappa_n \frac{(2\pi i)^n}{(n-1)!} \mathbb{Z}$ on $\widehat{\mathbb{C}}_{-+}$ and $\widehat{\mathbb{C}}_{--}$.*

2.1. Primitives for ω_n . Let Log denote the main branch of logarithm (argument in $(-\pi, \pi]$) and let Li_k denote the main branch of polylogarithm ($\text{Li}_1(z) = -\text{Log}(1-z)$, $\text{Li}_k(z) = \int_0^1 \frac{\text{Li}_{k-1}(tz)}{t} dt$). We may uniquely write each element (u, v) in $\widehat{\mathbb{C}}_{\pm}$ in the form

$$(2.4) \quad \langle z; p, q \rangle_{\epsilon_1, \epsilon_2} := (\text{Log}(\epsilon_1 z) + 2p\pi i, \text{Log}(\epsilon_2(1-z)) + 2q\pi i).$$

For $z \in \mathbb{C} \setminus \{0, 1\}$ and an integer q let

$$(2.5) \quad \mathcal{L}i_k(z; q) = \text{Li}_k(z) - \frac{2q\pi i}{(k-1)!} \text{Log}(z)^{k-1}.$$

Theorem 2.2 (Proof in Section 2.5.1). *Let $(u, v) = \langle z; p, q \rangle_{\epsilon_1, \epsilon_2}$ as in (2.4). The function*

$$(2.6) \quad \widehat{\mathcal{L}}_n(u, v) = \sum_{r=0}^{n-1} \frac{(-1)^r}{r!} \mathcal{L}i_{n-r}(z; q) u^r - \frac{(-1)^n}{n!} u^{n-1} v$$

is holomorphic and well defined modulo $\frac{(2\pi i)^n}{(n-1)!}$ for $(u, v) \in \widehat{\mathbb{C}}_{++}$ or $\widehat{\mathbb{C}}_{+-}$ and modulo $\kappa_n \frac{(2\pi i)^n}{(n-1)!}$ for $(u, v) \in \widehat{\mathbb{C}}_{-+}$. It is a primitive for ω_n , i.e. $d\widehat{\mathcal{L}}_n = \omega_n$.

Remark 2.3. On $\widehat{\mathbb{C}}_{--}$ the function (2.6) is only defined modulo $\frac{(\pi i)^n}{(n-1)!}$, and in order to obtain a primitive defined modulo $\kappa_n \frac{(2\pi i)^n}{(n-1)!}$, one must modify it by multiples of $\frac{(\pi i)^n}{(n-1)!}$. We refer to Section 2.5 for details.

Remark 2.4. The map $\widehat{\mathcal{L}}_2$ equals $R + \frac{\pi^2}{6}$ modulo π^2 , where R is Neumann's polylogarithm [Neu04].

2.2. Inversions and order 3 symmetries in low degree. It is well known that the polylogarithm \mathcal{L}_n in (1.2) satisfies the functional equations

$$(2.7) \quad \begin{aligned} \mathcal{L}_n(z) + (-1)^n \mathcal{L}_n(z^{-1}) &= 0, \\ \mathcal{L}_3(z) + \mathcal{L}_3\left(\frac{1}{1-z}\right) + \mathcal{L}_3(1-z^{-1}) &= \zeta(3), \quad \mathcal{L}_2(z) - \mathcal{L}_2\left(\frac{1}{1-z}\right) = 0. \end{aligned}$$

Consider the holomorphic maps

$$(2.8) \quad \tau: \widehat{\mathbb{C}}_{\pm} \rightarrow \widehat{\mathbb{C}}_{\pm}, \quad (u, v) \mapsto (-u, v-u), \quad \sigma: \widehat{\mathbb{C}}_{\pm} \rightarrow \widehat{\mathbb{C}}_{\pm}, \quad (u, v) \mapsto (-v, u-v).$$

One easily checks that τ and σ have order 2 and 3, respectively, and that they are lifts of the maps $\mathbb{C} \setminus \{0, 1\} \rightarrow \mathbb{C} \setminus \{0, 1\}$ given by $z \mapsto z^{-1}$ and $z \mapsto \frac{1}{1-z}$, respectively. We stress that τ and σ are not defined on $\widehat{\mathbb{C}}$. An elementary calculation shows that ($*$ denotes pullback of forms)

$$(2.9) \quad \tau^*(\omega_n) = -(-1)^n \omega_n, \quad \omega_3 + \sigma^* \omega_3 + \sigma^*(\sigma^*(\omega_3)) = 0, \quad \sigma^*(\omega_2) = \omega_2.$$

This implies that the functions

$$(2.10) \quad \widehat{\mathcal{L}}_n(u, v) + (-1)^n \widehat{\mathcal{L}}_n(-u, v - u), \\ \widehat{\mathcal{L}}_3(u, v) + \widehat{\mathcal{L}}_3(-v, u - v) + \widehat{\mathcal{L}}_3(v - u, -u), \quad \widehat{\mathcal{L}}_2(u, v) - \widehat{\mathcal{L}}_2(-v, u - v),$$

are locally constant.

Proposition 2.5. For any $(u, v) \in \mathbb{C}_{-+}$ we have

$$(2.11) \quad \widehat{\mathcal{L}}_n(u, v) + (-1)^n \widehat{\mathcal{L}}_n(-u, v - u) = (2^n - 2)(\pi i)^n \frac{B_n}{n!} \in \mathbb{C}/\kappa_n \frac{(2\pi i)^n}{(n-1)!} \mathbb{Z}.$$

In particular, it is zero if n is odd.

Proof. Since the left-hand side is constant on $\widehat{\mathbb{C}}_{-+}$ it is enough to consider $(u, v) = (0, \text{Log}(2)) \in \widehat{\mathbb{C}}_{-+}$. For this point $\tau(u, v) = (u, v)$, which proves the result for odd n . Since $\text{Li}_n(1) = \zeta(n)$ (this follows from (1.1)) it follows from the formula $\text{Li}_n(z) + \text{Li}_n(-z) = 2^{1-n} \text{Li}_n(z^2)$ [Lew91, p. 29] that $\text{Li}_n(-1) = -(1 - 2^{1-n})\zeta(n)$. When n is even, $\zeta(n) = (-1)^{\frac{n}{2}+1} \frac{B_n(2\pi)^n}{2n!}$, so we have

$$(2.12) \quad 2\widehat{\mathcal{L}}_n(0, \log(2)) = 2\text{Li}_n(-1) = -2(1 - 2^{1-n})\zeta(n) = (2^n - 2)(\pi i)^n \frac{B_n}{n!}.$$

This concludes the proof. \square

Corollary 2.6. When n is even, the order of (2.11) is the order of the torsion in $H_{\mathcal{M}}^1(\mathbb{Z}, \mathbb{Z}(n))$.

Proof. By the Staudt-Clausen formula for the denominator of even Bernoulli numbers, we see that the order of (2.11) is equal to the denominator of $\frac{B_n}{2n}$. The K -groups $K_{2n-1}(\mathbb{Z})$ are given in terms of integers $w_n(\mathbb{Q})$ in [Wei05, Thm. 1], and it follows from the exact sequences in [Lev, Thm 14.10, Rm. 14.11], relating $K_{2n-1}(\mathcal{O}_S)$ to $H_{\mathcal{M}}^1(\mathcal{O}_S, \mathbb{Z}(n))$ for number rings \mathcal{O}_S , that the torsion in $H_{\mathcal{M}}^1(\mathbb{Z}, \mathbb{Z}(n))$ has order $w_n(\mathbb{Q})$ for any n . The result now follows from the fact $w_n(\mathbb{Q})$ is the denominator of $\frac{B_n}{2n}$ when n is even. See also [Spi] for a table of $H_{\mathcal{M}}^p(\mathbb{Z}, \mathbb{Z}(q))$. \square

Remark 2.7. Proposition 2.5 also holds for $(u, v) \in \widehat{\mathbb{C}}_{--}$. For the other 2 components the right-hand side is $-(2\pi i)^n \frac{B_n}{n!}$ modulo $\frac{(2\pi i)^n}{(n-1)!}$. The order of this is half the order of $H_{\mathcal{M}}^1(\mathbb{Z}, \mathbb{Z}(n))$.

Lemma 2.8. For (u, v) in $\widehat{\mathbb{C}}_{++}$, $\widehat{\mathbb{C}}_{+-}$ or $\widehat{\mathbb{C}}_{-+}$ we have

$$(2.13) \quad \widehat{\mathcal{L}}_3(u, v) + \widehat{\mathcal{L}}_3(-v, u - v) + \widehat{\mathcal{L}}_3(v - u, -u) = \zeta(3) \pmod{4\pi^3 i}.$$

Proof. It is enough to check this for $(u, v) \in \widehat{\mathbb{C}}_{++}$. If (u, v) is a lift of -1 , then $(-v, u - v)$ and $(v - u, -u)$ are lifts of $\frac{1}{2}$ and 2, respectively. The result now follows from the formulas

$$(2.14) \quad \text{Li}_3(-1) = -\frac{3}{4}\zeta(3), \quad \text{Li}_3\left(\frac{1}{2}\right) = \frac{7}{8}\zeta(3) - \frac{\pi^2}{12}\log(2) + \frac{1}{6}\log(2)^3, \\ \text{Li}_3(2) = \frac{7}{8}\zeta(3) + \frac{\pi^2}{4}\log(2)^2 - \frac{\pi}{2}i\log(2)^2.$$

which can be found in [Lew81, A.2.6]. We leave the details of the computation to the reader. \square

Lemma 2.9. We have

$$(2.15) \quad \widehat{\mathcal{L}}_2(u, v) - \widehat{\mathcal{L}}_2(-v, u - v) = -\frac{\pi^2}{6} \pmod{\frac{\pi^2}{2}}.$$

Proof. This is proved using elementary properties of the dilogarithm. We omit the details. \square

2.3. Relationship between $\widehat{\mathcal{L}}_n$ and \mathcal{L}_n . For $(u, v) \in \mathbb{C}^2$, let

$$(2.16) \quad \det(u \wedge v) = \operatorname{Re}(u) \operatorname{Im}(v) - \operatorname{Im}(u) \operatorname{Re}(v).$$

The result below relates $\widehat{\mathcal{L}}_n$ to \mathcal{L}_n . It is a generalization of [DZ06, Prop. 4.6] for $n = 2$.

Theorem 2.10 (Proof in Section 6). *There exist rational numbers $c_{i,j}$ and $d_{i,j}$ such that*

$$(2.17) \quad \begin{aligned} \mathfrak{R}_n(\widehat{\mathcal{L}}_n(u, v)) - \mathcal{L}_n(r(u, v)) &= \sum_{s=1}^{n-2} \left(\mathfrak{R}_{n-s}(\widehat{\mathcal{L}}_{n-s}(u, v)) \sum_{i=0}^s c_{i,s-i} \operatorname{Re}(u)^i \operatorname{Im}(u)^{s-i} \right) + \\ &\det(u \wedge v) \sum_{i=0}^{n-2} d_{i,n-2-i} \operatorname{Re}(u)^i \operatorname{Im}(u)^{n-2-i}. \end{aligned}$$

Explicit formulas for $c_{i,j}$ and $d_{i,j}$ are given in Section 6. For example, we have

$$(2.18) \quad \begin{aligned} \operatorname{Im}(\widehat{\mathcal{L}}_2(u, v)) - \mathcal{L}_2(r(u, v)) &= -\frac{1}{2} \det(u \wedge v) \\ \operatorname{Re}(\widehat{\mathcal{L}}_3(u, v)) - \mathcal{L}_3(r(u, v)) &= \operatorname{Im}(\widehat{\mathcal{L}}_2(u, v)) \operatorname{Im}(u) + \frac{1}{6} \det(u \wedge v) \\ \operatorname{Im}(\widehat{\mathcal{L}}_4(u, v)) - \mathcal{L}_4(r(u, v)) &= -\operatorname{Re}(\widehat{\mathcal{L}}_3(u, v)) \operatorname{Im}(u) + \frac{1}{6} (\operatorname{Re}(u)^2 + 3 \operatorname{Im}(u)^2) \operatorname{Im}(\widehat{\mathcal{L}}_2(u, v)) + \\ &\frac{1}{24} \det(u \wedge v) (\operatorname{Re}(u)^2 + \operatorname{Im}(u)^2). \end{aligned}$$

Remark 2.11. This result is the key to proving that $\mathfrak{R}_n \circ \widehat{\mathcal{L}}_n$ agrees with $\mathcal{L}_n \circ r$ on $H^1(\widehat{\Gamma}(\mathbb{C}, n))$.

2.4. Limiting behavior. We have $\mathcal{L}_n(0) = 0$, but since 0 has no lift, there is no obvious analogue for $\widehat{\mathcal{L}}_n$.

Lemma 2.12. Let $z_k \in \mathbb{C} \setminus \{0, 1\}$ be a sequence with $z_k \rightarrow 0$, and let $\{p_k\} \subset \mathbb{Z}$ be a bounded sequence. Letting $k \rightarrow \infty$ we have

$$(2.19) \quad \widehat{\mathcal{L}}_n(\operatorname{Log}(z_k) + p_k \pi i, \operatorname{Log}(1 - z_k)) \rightarrow 0, \quad \widehat{\mathcal{L}}_n(\operatorname{Log}(1 - z_k), \operatorname{Log}(z_k) + 2p_k \pi i) \rightarrow \zeta(n)$$

Proof. By L'Hospital's rule, the sequence $\operatorname{Li}_s(z_k) \operatorname{Log}(z_k)^r$ tends to 0 for any positive integers r and s . This proves the first limit. The second limit follows from the fact that $\operatorname{Li}_n(1) = \zeta(n)$. \square

Remark 2.13. The derivative only determines $\widehat{\mathcal{L}}_n$ up to an integration constant. Lemma 2.12 determines the constant on $\widehat{\mathbb{C}}_{++}$ and $\widehat{\mathbb{C}}_{-+}$. The constant is then specified on $\widehat{\mathbb{C}}_{+-}$ by Remark 2.7, which also specifies the constant modulo 2-torsion on $\widehat{\mathbb{C}}_{--}$ when n is even.

2.5. The polylogarithm formula for $\widehat{\mathcal{L}}_n$. We now prove that the 1-forms ω_n have integral periods and that our formula for $\widehat{\mathcal{L}}_n$ is a primitive (Theorems 2.1 and 2.2). This is straightforward, but fairly technical, so we present detailed arguments.

We first give concrete models for each of the four components of $\widehat{\mathbb{C}}_{\pm}$ following [Neu04, GZ07]. For signs ϵ_1 and ϵ_2 (regarded when convenient as elements of $\{-1, 1\}$), let

$$(2.20) \quad \mathbb{C}_{\epsilon_1 \epsilon_2}^{\text{cut}} = \mathbb{C} \setminus \{z \in \mathbb{R} \mid \epsilon_1 z \leq 0, \epsilon_2(1 - z) \leq 0\}.$$

Note that $\mathbb{C}_{--}^{\text{cut}}$ is disconnected. Let

$$(2.21) \quad \overline{\mathbb{C}}_{\epsilon_1 \epsilon_2}^{\text{cut}} = \mathbb{C}_{\epsilon_1 \epsilon_2}^{\text{cut}} \cup \{z \pm 0i \mid z \in \mathbb{R}, \epsilon_1 z < 0, \epsilon_2(1-z) < 0\}.$$

The functions Li_k and Log extend continuously to $\overline{\mathbb{C}}_{\epsilon_1 \epsilon_2}^{\text{cut}}$. Define $\widehat{\mathbb{C}}_{\epsilon_1 \epsilon_2}$ to be the Riemann surface obtained from $\overline{\mathbb{C}}_{\epsilon_1 \epsilon_2}^{\text{cut}} \times \mathbb{Z}^2$ as the quotient by the relations

$$(2.22) \quad \begin{aligned} (z + 0i, p, q) &\sim (z - 0i, p + \epsilon_1, q) && \text{if } \epsilon_1 z < 0, \quad \epsilon_2(1-z) > 0 \\ (z + 0i, p, q) &\sim (z - 0i, p, q - \epsilon_2) && \text{if } \epsilon_1 z > 0, \quad \epsilon_2(1-z) < 0 \\ (z + 0i, p, q) &\sim (z - 0i, p + \epsilon_1, q - \epsilon_2) && \text{if } \epsilon_1 z < 0, \quad \epsilon_2(1-z) < 0. \end{aligned}$$

An equivalence class is denoted by $\langle z; p, q \rangle_{\epsilon_1 \epsilon_2}$. The map

$$(2.23) \quad \langle z; p, q \rangle_{\epsilon_1 \epsilon_2} \mapsto (\text{Log}(\epsilon_1 z) + 2p\pi i, \text{Log}(\epsilon_2(1-z)) + 2q\pi i)$$

identifies $\widehat{\mathbb{C}}_{\epsilon_1 \epsilon_2}$ with the appropriate component of $\widehat{\mathbb{C}}_{\pm}$.

2.5.1. *Definition of $\widehat{\mathcal{L}}_n$.* We begin with the definition of a map \widehat{L}_n , which agrees with $\widehat{\mathcal{L}}_n$ except on $\widehat{\mathbb{C}}_{--}$. Consider the map

$$(2.24) \quad \widehat{L}_n: \overline{\mathbb{C}}_{\epsilon_1 \epsilon_2}^{\text{cut}} \times \mathbb{Z}^2 \rightarrow \mathbb{C}$$

taking $\langle z; p, q \rangle_{\epsilon_1 \epsilon_2}$ to

$$(2.25) \quad \sum_{r=0}^{n-1} \frac{(-1)^r}{r!} \mathcal{L}i_{n-r}(z; q) \text{Log}(\epsilon_1 z; p)^r - \frac{(-1)^n}{n!} \text{Log}(\epsilon_1 z; p)^{n-1} \text{Log}(\epsilon_2(1-z); q),$$

where

$$(2.26) \quad \mathcal{L}i_k(z; q) = \text{Li}_k(z) - \frac{2q\pi i}{(k-1)!} \text{Log}(z)^{k-1}, \quad \text{Log}(z; p) = \text{Log}(z) + 2p\pi i.$$

We wish to show that \widehat{L}_n descends to a holomorphic function on $\widehat{\mathbb{C}}_{\epsilon_1 \epsilon_2}$. For $z \in \mathbb{R} \setminus \{0, 1\}$, let $z_{\pm} = z \pm 0i$, and let

$$(2.27) \quad \Delta_{\epsilon_1 \epsilon_2}(z, p, q) = \frac{(n-1)!}{(2\pi i)^n} \begin{cases} \widehat{L}_n(z_+, p, q) - \widehat{L}_n(z_-, p + \epsilon_1, q) & \text{if } \epsilon_1 z < 0, \quad \epsilon_2(1-z) > 0 \\ \widehat{L}_n(z_+, p, q) - \widehat{L}_n(z_-, p, q - \epsilon_2) & \text{if } \epsilon_1 z > 0, \quad \epsilon_2(1-z) < 0 \\ \widehat{L}_n(z_+, p, q) - \widehat{L}_n(z_-, p + \epsilon_1, q - \epsilon_2) & \text{if } \epsilon_1 z < 0, \quad \epsilon_2(1-z) < 0 \\ 0 & \text{if } \epsilon_1 z > 0, \quad \epsilon_2(1-z) > 0. \end{cases}$$

Clearly, $\Delta_{\epsilon_1 \epsilon_2}(z, p, q)$ only depends on the interval I (either $(-\infty, 0)$, $(0, 1)$, or $(1, \infty)$) where z belongs. We denote it by $\Delta_{\epsilon_1 \epsilon_2}^I(p, q)$ accordingly. Let

$$(2.28) \quad \delta(p, n) = (-1)^n((p-1)^{n-1} - p^{n-1}).$$

Lemma 2.14. We have

$$(2.29) \quad \begin{aligned} \Delta_{++}^{(-\infty, 0)}(p, q) &= q\delta(p+1, n), & \Delta_{++}^{(0, 1)}(p, q) &= \Delta_{++}^{(1, \infty)}(p, q) = 0 \\ \Delta_{-+}^{(-\infty, 0)}(p, q) &= q\delta(p + \frac{1}{2}, n), & \Delta_{-+}^{(0, 1)}(p, q) &= \Delta_{-+}^{(1, \infty)}(p, q) = 0 \\ \Delta_{+-}^{(-\infty, 0)}(p, q) &= (-p-1)^{n-1} + q\delta(p+1, n), & \Delta_{+-}^{(0, 1)}(p, q) &= \Delta_{+-}^{(1, \infty)}(p, q) = (-p)^{n-1} \\ \Delta_{--}^{(-\infty, 0)}(p, q) &= (-p - \frac{1}{2})^{n-1} + q\delta(p + \frac{1}{2}, n), & \Delta_{--}^{(0, 1)}(p, q) &= \Delta_{--}^{(1, \infty)}(p, q) = (\frac{1}{2} - p)^{n-1} \end{aligned}$$

Proof. Suppose $z < 0$. Then $\text{Li}_k(z_+) = \text{Li}_k(z_-)$ and we have

$$(2.30) \quad \widehat{L}_n(\langle z_+; p, q \rangle_{++}) - \widehat{L}_n(\langle z_-; p+1, q \rangle_{++}) = \\ - \frac{2q\pi i}{(n-1)!} \left(\sum_{r=0}^{n-1} \binom{n-1}{r} (\text{Log}(z_+))^{n-r-1} (-\text{Log}(z_+; p))^r \right) - \\ \sum_{r=0}^{n-1} \binom{n-1}{r} \text{Log}(z_-)^{n-r-1} (-\text{Log}(z_-; p+1))^r \Big) = \\ - \frac{2q\pi i}{(n-1)!} \left((-2p\pi i)^{n-1} - (-2(p+1)\pi i)^{n-1} \right) = \frac{(2\pi i)^n}{(n-1)!} q \delta(p+1, n).$$

This proves the first equality. Some of the other ones make use of the identity

$$(2.31) \quad \text{Li}_n(z_+) - \text{Li}_n(z_-) = \frac{2\pi i \text{Log}(z)^{n-1}}{(n-1)!}, \quad z \in (1, \infty)$$

but are otherwise similar. We leave their verification to the reader. \square

Corollary 2.15. \widehat{L}_n is holomorphic on $\widehat{\mathbb{C}}_{\pm}$ and defined modulo $\frac{(2\pi i)^n}{(n-1)!}$ on $\widehat{\mathbb{C}}_{++}$ and $\widehat{\mathbb{C}}_{+-}$, modulo $\kappa_n \frac{(2\pi i)^n}{(n-1)!}$ on $\widehat{\mathbb{C}}_{-+}$, and modulo $\frac{(\pi i)^n}{(n-1)!}$ on $\widehat{\mathbb{C}}_{--}$.

Proof. For $\widehat{\mathbb{C}}_{++}$, $\widehat{\mathbb{C}}_{+-}$ and $\widehat{\mathbb{C}}_{--}$ this follows immediately from Lemma 2.14. When expanding $\delta(p + \frac{1}{2}, n)$ one easily verifies that the greatest common divisor of the denominators is 2^{n-2} if n is even and $2^{n-3-\nu_2(n-1)}$ if n is odd. This proves the case $\widehat{\mathbb{C}}_{-+}$. \square

Lemma 2.16. \widehat{L}_n is a primitive for the one form ω_n defined in (2.2), i.e. $d\widehat{L}_n = \omega_n$.

Proof. By (1.1) one has $d\text{Li}_k(z) = \frac{\text{Li}_{k-1}(z)}{z} dz$, and it follows that $d\mathcal{L}i_k(z; q) = \frac{\mathcal{L}i_{k-1}(z; q)}{z} dz$. This holds for all $k \geq 0$ with the convention that $\mathcal{L}i_0(z; q) = \frac{z}{1-z}$. Letting $u = \text{Log}(\epsilon_1 z; p)$, $v = \text{Log}(\epsilon_2(1-z); q)$, $\widehat{L}_n = \widehat{L}_n(\langle z; p, q \rangle_{\epsilon_1 \epsilon_2})$, and $\mathcal{L}i_k = \mathcal{L}i_k(z; q)$ one has

$$(2.32) \quad d\widehat{L}_n = \sum_{r=0}^{n-1} \frac{(-1)^r}{r!} (\mathcal{L}i_{n-r-1} u^r + r \mathcal{L}i_{n-r} u^{r-1}) \frac{dz}{z} - \frac{(-1)^n}{n!} ((n-1)u^{n-2} v du + u^{n-1} dv) \\ = \frac{(-1)^{n-1}}{(n-1)!} u^{n-1} \mathcal{L}i_0 \frac{dz}{z} - \frac{(-1)^n}{n!} ((n-1)u^{n-2} v du + u^{n-1} dv) = \omega_n.$$

The second equality follows by telescoping, and the third from the fact that $\mathcal{L}i_0(z; q) \frac{dz}{z} = -dv$. \square

Theorem 2.17. *The form ω_n has periods in $\frac{(2\pi i)^n}{(n-1)!} \mathbb{Z}$ on $\widehat{\mathbb{C}}_{++}$ and $\widehat{\mathbb{C}}_{+-}$ and in $\kappa_n \frac{2\pi i}{(n-1)!}$ on $\widehat{\mathbb{C}}_{-+}$ and $\widehat{\mathbb{C}}_{-+}$, where κ_n is defined in (2.3).*

Proof. The commutator subgroup of $\pi_1(\mathbb{C} \setminus \{0, 1\})$ is generated by the loops $\gamma_{k,l} = a^k b^l a^{-k} b^{-l}$ where a is a loop going counterclockwise around 0 and b is a loop going clockwise around 1. It is thus enough to compute the integral of ω_n along a lift of $\gamma_{k,l}$. Since \widehat{L}_n is a primitive, this is always of the form $\frac{2\pi i}{(n-1)!} A_{k,l}$ where $A_{k,l}$ is an integral linear combination of terms $\Delta_{\epsilon_1 \epsilon_2}^I(r, s)$. One easily checks that when $\epsilon_1 = 1$, $A_{k,l} \in \mathbb{Z}$, and when $\epsilon_1 = -1$, $A_{k,l} \in \kappa_n \mathbb{Z}$. For example, if $k = l = 1$ we have

$$(2.33) \quad A_{1,1} = -\Delta_{\epsilon_1 \epsilon_2}^{(-\infty, 0)}(p, q) + \Delta_{\epsilon_1 \epsilon_2}^{(0, 1)}(p+1, q) - \Delta_{\epsilon_1 \epsilon_2}^{(1, \infty)}(p+1, q) + \\ \Delta_{\epsilon_1 \epsilon_2}^{(-\infty, 0)}(p, q-1) - \Delta_{\epsilon_1 \epsilon_2}^{(0, 1)}(p, q-1) + \Delta_{\epsilon_1 \epsilon_2}^{(1, \infty)}(p, q),$$

which equals $-\delta(p+1, n) \in \mathbb{Z}$ when $\epsilon_1 = 1$ and $-\delta(p + \frac{1}{2}, n) \in \kappa_n \mathbb{Z}$ when $\epsilon_1 = -1$. \square

2.5.2. *Modifying \widehat{L}_n on $\widehat{\mathbb{C}}_{--}$.* We now define $\widehat{\mathcal{L}}_n$ to be \widehat{L}_n except on $\widehat{\mathbb{C}}_{--}$, where we define it as (with the convention that $\text{Im}(z+0i) > 0$ and $\text{Im}(z-0i) < 0$ for $z \in \mathbb{R}$)

$$(2.34) \quad \widehat{\mathcal{L}}_n = \widehat{L}_n + \frac{(2\pi i)^n}{(n-1)!} \begin{cases} -(-p - \frac{1}{2})^{n-1} + (-\frac{1}{2})^{n-1} & \text{if } \text{Im}(z) > 0 \\ (-\frac{1}{2})^{n-1} & \text{if } \text{Im}(z) < 0. \end{cases}$$

This is well defined since the imaginary part is never zero on $\overline{\mathbb{C}}_{--}^{\text{cut}}$. The fact that $\widehat{\mathcal{L}}_n$ is defined modulo $\kappa_n \frac{2\pi i}{(n-1)!}$ on $\widehat{\mathbb{C}}_{--}$ is an easy consequence of Lemma 2.14.

3. $\widehat{\mathcal{L}}_n$ RELATIONS AND THE SYMBOL MAP

To motivate our treatment of $\widehat{\mathcal{L}}_n$ relations we begin with a review of the so-called *symbol* map. The new idea is that in a certain sense, the vanishing of the symbol map is equivalent to the vanishing of a certain 1-form, which is a multiple of the 1-form $d\widehat{\mathcal{L}}_n$ (see Proposition 3.10).

3.1. **The symbol map.** The *symbol map* is the map ($[0]$, $[1]$, and $[\infty]$ map to 0)

$$(3.1) \quad \text{symb}_n: \mathbb{Z}[P_F^1] \rightarrow \wedge^2(F^*) \otimes \text{Sym}^{n-2}(F^*), \quad [z] \mapsto z \wedge (1-z) \otimes z^{\otimes(n-2)}.$$

It follows from Goncharov's definition of $\Gamma(F, n)$ that symb_n factors through $\mathcal{B}_n(F)$, and that if $\beta \in \mathbb{Z}[P_F^1]$ satisfies $\delta(\beta) = 0 \in \mathcal{B}_{n-1} \otimes F^*$, then $\text{symb}_n(\beta) = 0$. The converse is false.

The two results below link the symbol map to functional relations for \mathcal{L}_n .

Theorem 3.1 (Zagier [Zag91, Prop. 3], Goncharov [Gon94, Thm. 1.17]). *Let $\beta \in \mathbb{Z}[P_{\mathbb{C}(t)}^1]$. If $\text{symb}_n(\beta) = 0$, then the $\mathcal{L}_n(\beta)$ (regarded as a function in t) is constant. Moreover, if $\mathcal{L}_n(\beta)$ is identically 0, then $\delta(\beta) = 0$, so $\beta(t) \in \mathbb{Z}[P_{\mathbb{C}}^1]$ is constant in $\mathcal{B}_n(\mathbb{C})$.*

The following discussion serves mainly to motivate later definitions. Let $\beta = \sum_{i=1}^M r_i [z_i] \in \mathbb{Z}[F(t)]$ and suppose that for $a_1, \dots, a_N \subset F(t)^*$ and integers k_{ji} and l_{ji} we have

$$(3.2) \quad z_i = a_1^{k_{1i}} \cdots a_N^{k_{Ni}}, \quad 1 - z_i = a_1^{l_{1i}} \cdots a_N^{l_{Ni}}.$$

For each integer $0 < l < n-1$ and each multisubset (elements need not be distinct) $J = \{j_1, \dots, j_l\}$ of $\{1, \dots, N\}$ define

$$(3.3) \quad \pi_J = \sum_{i=1}^M r_i k_{j_1 i} \cdots k_{j_l i} [z_i] \in \mathbb{Z}[F(t)].$$

The following is an easy induction argument using that $z_i^{\otimes l} = \sum_{|J|=l} k_{j_1 i} \cdots k_{j_l i} a_{j_1} \otimes \cdots \otimes a_{j_l}$.

Proposition 3.2. Suppose $\text{symb}_n(\beta) = 0$. If for some t_0 , the elements $\pi_J(t_0)$ are zero in $\mathcal{B}_{n-l}(F)$ for all J with $|J| = l$, then $\delta(\beta) = 0 \in \mathcal{B}_{n-1}(F(t)) \otimes F(t)^*$. In particular, it follows that

$$(3.4) \quad \beta(t) - \beta(t_0) \in R_n(F) \text{ for all } t.$$

3.2. Our setup. Let a_i and \tilde{a}_i be formal variables. We think of \tilde{a}_i as a logarithm of a_i . Consider the polynomial rings

$$(3.5) \quad S = \mathbb{Z}[a_1^{\pm 1}, a_2^{\pm 1}, \dots], \quad \tilde{S} = \mathbb{Z}[\tilde{a}_1, \tilde{a}_2, \dots].$$

Let $\tilde{S}_k \subset \tilde{S}$ denote the group of homogeneous polynomials of degree k , and let U denote the free multiplicative group on the a_i . We have a canonical group homomorphism

$$(3.6) \quad \pi: \tilde{S}_1 \rightarrow U, \quad \tilde{a}_i \mapsto a_i.$$

We shall consider elements $\alpha \in \mathbb{Z}[\tilde{S}_1 \times \tilde{S}_1]$. Each such can be canonically written as $\sum_{i=1}^M r_i(u_i, v_i)$ where $r_i \in \mathbb{Z}$ and

$$(3.7) \quad u_i = \sum_{j=1}^N k_{ji} \tilde{a}_j \in \tilde{S}_1, \quad v_i = \sum_{j=1}^N l_{ji} \tilde{a}_j \in \tilde{S}_1$$

Convention 3.3. For $\alpha \in \mathbb{Z}[\tilde{S}_1 \times \tilde{S}_1]$ we always define u_i and v_i as in (3.7). We similarly define $r = (r_1, \dots, r_M)$, $K = \{k_{ji}\}$, $L = \{l_{ji}\}$, and

$$(3.8) \quad z_i = \pi(u_i) = \prod_{j=1}^N a_j^{k_{ji}} \in U, \quad w_i = \pi(v_i) = \prod_{j=1}^N a_j^{l_{ji}} \in U.$$

3.3. Realizations. Let $\alpha \in \mathbb{Z}[\tilde{S}_1 \times \tilde{S}_1]$.

Definition 3.4. The *realization scheme* for α is the scheme X_α defined by the *realization equations*

$$(3.9) \quad \prod_{j=1}^N a_j^{k_{ji}} + \prod_{j=1}^N a_j^{l_{ji}} - 1 \in \mathbb{Z}[a_1^{\pm 1}, \dots, a_N^{\pm 1}], \quad i = 1, \dots, M.$$

More precisely $X_\alpha = \text{Spec}(\mathbb{Z}[a_1^{\pm 1}, \dots, a_N^{\pm 1}]/I_\alpha)$ where I_α is the ideal generated by the realization equations. For a field F we let $X_\alpha(F) = \text{Spec}(F[a_1^{\pm 1}, \dots, a_N^{\pm 1}]/I_\alpha)$.

Note that X_α only depends on K and L .

Convention 3.5. By a point p in $X_\alpha(F)$ we always mean a rational point, i.e. a ring homomorphism $p: \mathbb{Z}[a_1^{\pm 1}, \dots, a_N^{\pm 1}] \rightarrow F$ killing the realization ideal. We also assume that p is smooth. We extend p to S by mapping a_i to 1 for $i > N$. Then p restricts to a group homomorphism $p_U: U \rightarrow \mathbb{C}^*$.

Definition 3.6. Let p be a point in $X_\alpha(\mathbb{C})$. A *lift* of p is a homomorphism $\tilde{p}: \tilde{S}_1 \rightarrow \mathbb{C}$ lifting p_U in the sense that $\text{exp} \circ \tilde{p} = p_U \circ \pi$. A point together with a lift is called a *log-point*.

A log-point \tilde{p} in $X_\alpha(\mathbb{C})$ determines elements

$$(3.10) \quad \tilde{p}(\alpha) = \sum_{i=1}^M r_i[(\tilde{p}(u_i), \tilde{p}(v_i))] \in \mathbb{Z}[\widehat{\mathbb{C}}], \quad p(\alpha) = \sum_{i=1}^M r_i[z_i] \in \mathbb{Z}[\mathbb{C} \setminus \{0, 1\}] \subset \mathbb{Z}[P_{\mathbb{C}}^1]$$

such that $r(\tilde{p}(\alpha)) = p(\alpha)$, where r is the covering $\widehat{\mathbb{C}} \rightarrow \mathbb{C} \setminus \{0, 1\}$. We shall think of α as a purely symbolic representation of a Bloch group element, and we refer to $\tilde{p}(\alpha)$ and $p(\alpha)$ as *realizations* of α in $\mathbb{Z}[\widehat{\mathbb{C}}]$ and $\mathbb{Z}[P_{\mathbb{C}}^1]$, respectively. If $Y \subset X_\alpha(\mathbb{C})$ is a smooth submanifold and \tilde{p}_Y is a family of log-points Y , (3.10) gives rise to maps

$$(3.11) \quad \tilde{p}_Y(\alpha): Y \rightarrow \mathbb{Z}[\widehat{\mathbb{C}}], \quad p_Y(\alpha): Y \rightarrow \mathbb{Z}[\mathbb{C} \setminus \{0, 1\}].$$

In particular, $\widehat{\mathcal{L}}_n \circ \widetilde{p}_Y(\alpha)$ and $\mathcal{L}_n \circ p_Y(\alpha)$ are functions on Y . If the logarithms of the coordinates are smooth, we say that \widetilde{p}_Y is a *smooth family of log-points over Y* . If so, $\widehat{\mathcal{L}}_n \circ \widetilde{p}_Y(\alpha)$ is smooth.

3.4. Differential $\widehat{\mathcal{L}}_n$ relations and the symbol. For an integer $k > 0$ let $\Omega_k^1(\widetilde{S})$ denote the group of 1-forms on S of degree k (finite formal sums of terms $f_I d\widetilde{a}_i$ where f_I is a degree k monomial).

Definition 3.7. We say that α is a *differential $\widehat{\mathcal{L}}_n$ relation* if

$$(3.12) \quad w_n(\alpha) = \sum_{i=1}^M r_i u_i^{n-2} (u_i dv_i - v_i du_i)$$

is 0 in $\Omega_{n-1}^1(\widetilde{S})$.

Proposition 3.8. Suppose α is a differential $\widehat{\mathcal{L}}_n$ relation. For any smooth family \widetilde{p}_Y of log-points over a connected $Y \subset X_\alpha(\mathbb{C})$ the function $\widehat{\mathcal{L}}_n \circ \widetilde{p}_Y(\alpha)$ is constant on Y .

Proof. This follows from the fact that $d\widehat{\mathcal{L}}_n(u, v) = (-1)^{n-1} \frac{1}{n!} u^{n-2} (u dv - v du)$. \square

Remark 3.9. Although any differential $\widehat{\mathcal{L}}_n$ relation with $\dim(X_\alpha(\mathbb{C})) > 0$ provides local $\widehat{\mathcal{L}}_n$ relations, the value of the constant $\widehat{\mathcal{L}}_n \circ \widetilde{p}_Y(\alpha)$ may depend dramatically on the choice of smooth logarithms (see Example 3.21 for an example).

The result below relates our notion of differential $\widehat{\mathcal{L}}_n$ relation to the vanishing of the symbol.

Proposition 3.10. The 1-form $w_n(\alpha)$ vanishes in $\Omega_{n-1}^1(\widetilde{S})$ if and only if the element $A = \sum_{i=1}^M r_i (z_i \wedge w_i) \otimes z_i^{n-2}$ in $\wedge^2(U) \otimes \text{Sym}^{n-2}(U)$ has order 1 or 2.

Proof. Consider the composition

$$(3.13) \quad \wedge^2(U) \otimes \text{Sym}^{n-2}(U) \xrightarrow{\cong} \wedge^2(\widetilde{S}_1) \otimes \text{Sym}^{n-2}(\widetilde{S}_1) \xrightarrow{\cong} \wedge^2(\widetilde{S}_1) \otimes \widetilde{S}_{n-2} \hookrightarrow \Omega_{n-1}^1(\widetilde{S}).$$

The left isomorphism is induced by the canonical map $U \rightarrow \widetilde{S}_1$ taking a_i to \widetilde{a}_i , the middle isomorphism is induced by the canonical identification $\text{Sym}^k(\widetilde{S}_1) \cong \widetilde{S}_k$, and the right map takes $\widetilde{a}_i \wedge \widetilde{a}_j \otimes f$ to $f(\widetilde{a}_i d\widetilde{a}_j - \widetilde{a}_j d\widetilde{a}_i)$. The right map is injective modulo 2-torsion, and the composition takes A to $w_n(\alpha)$. This proves the result. \square

Corollary 3.11. Suppose $\beta = \sum_{i=1}^M r_i [z_i]$ is in the kernel of symb_n , and suppose that there are multiplicatively independent functions $a_j \in \mathbb{C}(x)^*$ such that (3.2) holds. Any choice of local log branches of the a_i determines a local $\widehat{\mathcal{L}}_n$ relation

$$(3.14) \quad \sum_{i=1}^M \widehat{\mathcal{L}}_n(u_i, v_i) = c,$$

where c is a constant, u_i and v_i are given by (3.7) with $\widetilde{a}_i = \log(a_i)$.

Remark 3.12. There are many examples where (3.2) only holds up to signs. For example the element $[x] + (-1)^n [x^{-1}] \in \mathcal{B}_n(\mathbb{C}(x))$ or $[x] + [\frac{1}{1-x}] + [1 - x^{-1}] \in \mathcal{B}_3(\mathbb{C}(x))$. The same is true for Goncharov's element $R(x, y, z)$ (see Conjecture 1.2 b)), Gangl's 931 term relation [Gan16] and many others. As we see in section 3.6 such \mathcal{L}_n relations have lifts to $\widehat{\mathcal{C}}_\pm$ (but not $\widehat{\mathcal{C}}$).

3.5. Examples. For notational convenience, we shall occasionally use *multiplicative notation* to denote elements in $\tilde{S}_1 \times \tilde{S}_1$, e.g. we write $(a_1 a_2, a_3 a_4)$ instead of $(\tilde{a}_1 + \tilde{a}_2, \tilde{a}_3 + \tilde{a}_4)$. We shall also occasionally denote the free variables by other symbols than a_i .

Example 3.13. The element $[(\tilde{a}_1, \tilde{a}_2)] + [(\tilde{a}_2, \tilde{a}_1)]$ is a differential $\widehat{\mathcal{L}}_2$ relation. Clearly, $X_\alpha(\mathbb{C}) = \mathbb{C} \setminus \{0, 1\}$ and a lift of a point $x \in X_\alpha(\mathbb{C})$ is a pair of complex numbers (u, v) , with $e^u = x$ and $e^v = 1 - x$. The corresponding relation is

$$(3.15) \quad \widehat{\mathcal{L}}_2(u, v) + \widehat{\mathcal{L}}_2(v, u) = \frac{\pi^2}{6} \in \mathbb{C}/4\pi^2\mathbb{Z},$$

which we regard as a lift of the relation $\mathcal{L}_2(x) + \mathcal{L}_2(1 - x) = 0$.

Example 3.14 (The lifted five term relation). The element

$$(3.16) \quad \alpha = [(a_1, a_3)] - [(a_2, a_4)] + \left[\left(\frac{a_2}{a_1}, \frac{a_5}{a_1}\right)\right] - \left[\left(\frac{a_2 a_3}{a_1 a_4}, \frac{a_5}{a_1 a_4}\right)\right] + \left[\left(\frac{a_3}{a_4}, \frac{a_5}{a_4}\right)\right]$$

is a differential $\widehat{\mathcal{L}}_2$ relation. One easily checks that $X_\alpha(\mathbb{C}) = \{(x, y) \in \mathbb{C} \setminus \{0, 1\} \mid x \neq y\}$ with

$$(3.17) \quad a_1 = x, \quad a_2 = y, \quad a_3 = 1 - x, \quad a_4 = 1 - y, \quad a_5 = x - y.$$

For each log-point \tilde{p} , the realization $\tilde{p}(\alpha)$ is an instance of Neumann's lifted five term relation [Zic15, Def. 3.2]. It thus follows (recall that Neumann's R equals $\widehat{\mathcal{L}}_2 - \frac{\pi^2}{6}$) that $\widehat{\mathcal{L}}_2(\tilde{p}(\alpha)) = \frac{\pi^2}{6}$.

Example 3.15 (The (inverted) lifted five term relations). By Example 3.13

$$(3.18) \quad \beta = -[(a_3, a_1)] + [(a_4, a_2)] - \left[\left(\frac{a_5}{a_1}, \frac{a_2}{a_1}\right)\right] + \left[\left(\frac{a_5}{a_1 a_4}, \frac{a_2 a_3}{a_1 a_4}\right)\right] - \left[\left(\frac{a_5}{a_4}, \frac{a_3}{a_4}\right)\right]$$

is also a differential $\widehat{\mathcal{L}}_2$ relation with $\widehat{\mathcal{L}}_2(\tilde{p}(\beta)) = 0 \in \mathbb{C}/4\pi^2\mathbb{Z}$ for each log-point \tilde{p} .

Example 3.16 (A 31 term relation). The free variables a_i with $i \in \{1, 2, 3\}$, and b_J with $\emptyset \neq J \subset \{1, 2, 3\}$ admit a natural cyclic \mathbb{Z}_3 action. The 31 term element

$$(3.19) \quad \alpha = [(a_1 a_2 a_3, b_{123})] + \sum_{\sigma \in \mathbb{Z}_3} \sigma A,$$

where A is given by

$$(3.20) \quad \begin{aligned} & [(a_1, b_1)] - \left[\left(\frac{b_1}{b_{12}}, \frac{a_1 b_2}{b_{12}}\right)\right] - \left[\left(\frac{b_1}{b_{13}}, \frac{a_1 b_3}{b_{13}}\right)\right] + \left[\left(\frac{b_1}{b_{123}}, \frac{a_1 b_{23}}{b_{123}}\right)\right] - [(a_1 a_2, b_{12})] + \\ & - \left[\left(\frac{a_1 b_2}{b_{12}}, \frac{b_1}{b_{12}}\right)\right] - \left[\left(\frac{a_1 b_3}{b_{13}}, \frac{b_1}{b_{13}}\right)\right] + \left[\left(\frac{b_1 b_{123}}{b_{12} b_{13}}, \frac{a_1 b_2 b_3}{b_{12} b_{13}}\right)\right] + \left[\left(\frac{a_1 a_2 b_3}{b_{123}}, \frac{b_{12}}{b_{123}}\right)\right] + \left[\left(\frac{a_1 b_2 b_3}{b_{12} b_{13}}, \frac{b_1 b_{123}}{b_{12} b_{13}}\right)\right] \end{aligned}$$

is a differential $\widehat{\mathcal{L}}_3$ relation. The points in $X_\alpha(F)$ may be identified with triples $(x_1, x_2, x_3) \in F^*$ all products of which are distinct from 1, with the identification $a_i = x_i$, $b_J = \prod_{j \in J} x_j$. We shall see later (Example 4.23) that when $F = \mathbb{C}$, the relation $\widehat{\mathcal{L}}_3(\tilde{p}(\alpha))$ is zero in $\mathbb{C}/\frac{(2\pi i)^3}{2}\mathbb{Z}$ for each log-point \tilde{p} .

Remark 3.17. After a change of variables $(x_1, x_2, x_3) = \left(\frac{y_2(1-y_1(1-y_3))}{1-y_3(1-y_2)}, y_3, \frac{1}{1-y_1(1-y_3)}\right)$ the element $p(\alpha)$ is an instance of Goncharov's 22 term element $R(y_1, y_2, y_3)$ up to instances of $[x] = [x^{-1}]$ and $[x] + [1 - x] + [1 - 1/x] = [1]$.

3.6. Realizations in $\widehat{\mathbb{C}}_{\pm}$. To define realizations of α in $\widehat{\mathbb{C}}_{\pm}$ we need additional data in the form of a *sign determination*, which is a vector $\mathcal{V} = ((\epsilon_{1,1}, \epsilon_{2,1}), \dots, (\epsilon_{1,M}, \epsilon_{2,M}))$ of sign pairs. Given such we have a realization scheme $X_{\alpha, \mathcal{V}}$ defined as in Definition 3.4, using the ideal generated by $\epsilon_{1,i}\pi(u_i) + \epsilon_{2,i}\pi(v_i) - 1$. If \tilde{p} is a lift of a point in $X_{\alpha, \mathcal{V}}$, we can define $\tilde{p}(\alpha) \in \mathbb{Z}[\widehat{\mathbb{C}}_{\pm}]$ and $p(\alpha) \in \mathbb{Z}[\mathbb{C} \setminus \{0, 1\}]$ as in (3.10) (but with $z_i = \epsilon_{1,i}\pi(u_i)$). Note that $(\tilde{p}(u_i), \tilde{p}(v_i))$ is in the component $\widehat{\mathbb{C}}_{\epsilon_{1,i}, \epsilon_{2,i}}$ of $\widehat{\mathbb{C}}_{\pm}$. We shall thus sometimes denote $(u_i, v_i) \in \tilde{S}_1 \times \tilde{S}_1$ by $(u_i, v_i)_{\epsilon_{1,i}, \epsilon_{2,i}}$.

Example 3.18. $[(\tilde{a}_1, \tilde{a}_2)_{-+}] + (-1)^n [(-\tilde{a}_1, \tilde{a}_2 - \tilde{a}_1)_{-+}]$ is a differential $\widehat{\mathcal{L}}_n$ relation. The corresponding relation is an instance of (2.11).

Example 3.19. $[(\tilde{a}_1, \tilde{a}_2)] + [(-\tilde{a}_2, \tilde{a}_1 - \tilde{a}_2)] + [(\tilde{a}_2 - \tilde{a}_1, -\tilde{a}_1)]$ is a differential $\widehat{\mathcal{L}}_3$ relation. The corresponding $\widehat{\mathcal{L}}_3$ relation for $\mathcal{V} = ((1, 1), (1, -1), (-1, 1))$ or any cyclic permutation of \mathcal{V} is an instance of (2.13).

Example 3.20 (Goncharov's 22 term relation). Consider free variables $\alpha_i, \beta_i, \gamma_i$ with $i \in \{1, 2, 3\}$ and an additional variable δ . Let \mathbb{Z}_3 act by cyclic permutation on α_i, β_i and γ_i and trivially on δ . The element

$$(3.21) \quad [(\alpha_1 \alpha_2 \alpha_3, \delta)_{-+}] + \sum_{\sigma \in \mathbb{Z}_3} \sigma(A),$$

where A is given by

$$(3.22) \quad \begin{aligned} & [(\alpha_i, \gamma_i)_{++}] + [(\beta_i, \alpha_i \gamma_{i-1})_{++}] - [(\frac{\alpha_{i-1}}{\beta_i}, \frac{\gamma_{i-1} \gamma_i}{\beta_i})_{++}] + [(\frac{\beta_i}{\alpha_{i-1} \alpha_i}, \frac{\gamma_i}{\alpha_{i-1} \alpha_i})_{+-}] + \\ & [(\frac{\alpha_i \beta_{i-1}}{\beta_{i+1}}, \frac{\gamma_{i+1} \beta_i}{\beta_{i+1}})_{++}] + [(\frac{\beta_i}{\alpha_i \beta_{i-1}}, \frac{\delta}{\alpha_i \beta_{i-1}})_{-+}] - [(\frac{\alpha_{i-1} \alpha_i \beta_{i+1}}{\beta_i}, \frac{\delta \gamma_i}{\beta_i})_{++}], \end{aligned}$$

is a differential $\widehat{\mathcal{L}}_3$ relation. The F points in $X_{\alpha, \mathcal{V}}$ may be identified with

$$(3.23) \quad \{(y_1, y_2, y_3) \in (F \setminus \{0, 1\})^3 \mid y_i(1 - y_{i-1}) \neq 1 \text{ (indices mod 3)}, \quad y_1 y_2 y_3 \neq -1\}$$

with $\alpha_i = y_i, \beta_i = 1 - y_i(1 - y_{i-1}), \gamma_i = 1 - y_i$, and $\delta = 1 + y_1 y_2 y_3$. For each point in $p \in X_{\alpha, \mathcal{V}}(\mathbb{C})$, $p(\alpha)$ is an instance of Goncharov's 22 term relation, so we have $\mathcal{L}_3(p(\alpha)) = 3\zeta(3)$ (see [Zag91, p. 428]). One can show that $\widehat{\mathcal{L}}_3(\tilde{p}(\alpha)) = 3\zeta(3)$ holds as well (see Remark 4.33).

In the previous examples the choice of lift made no difference to the value of $\widehat{\mathcal{L}}_n(\tilde{p}(\alpha))$. In other words, logarithms of a_i could be chosen independently and arbitrarily without affecting the $\widehat{\mathcal{L}}_n$ relation. The next example shows that this is not always the case.

Example 3.21. $\alpha = [(2\tilde{a}_1 + 2\tilde{a}_3, -\tilde{a}_1 + \tilde{a}_2)] - 2^{n-1} [(\tilde{a}_1 + \tilde{a}_3, -2\tilde{a}_1 + \tilde{a}_2 - \tilde{a}_3)]$ is a differential $\widehat{\mathcal{L}}_n$ relation. One checks that $X_{\alpha, \mathcal{V}}(\mathbb{C})$ is empty for $\mathcal{V} = ((1, 1), (1, 1))$, so there are no realizations in $\widehat{\mathbb{C}}$. For $\mathcal{V} = ((-1, 1), (1, 1))$, $X_{\alpha, \mathcal{V}}(\mathbb{C})$ is given by

$$(3.24) \quad a_1 = x, \quad a_2 = \frac{1}{2}(\omega + 1)x, \quad a_3 = \frac{\omega}{x}, \quad x \in \mathbb{C} \setminus \{0\}, \quad 2\omega^2 - \omega + 1 = 0.$$

For any log-point \tilde{p} , $p(\alpha) = [-\omega^2] - 2^{n-1}[\omega] \in \mathbb{Z}[\mathbb{C} \setminus \{0, 1\}]$, which only depends on ω . However, one easily checks (e.g. numerically) that different lifts give different values of $\widehat{\mathcal{L}}_n(\tilde{p}(\alpha))$ when $n > 2$. We shall not need the values, so we omit them.

3.7. Realizations for arbitrary fields. Fix a \mathbb{Z} -extension

$$(3.25) \quad 0 \rightarrow \mathbb{Z} \xrightarrow{\iota} E \xrightarrow{\pi} F^* \rightarrow 0$$

of F^* such that E is torsion free. If $F = \mathbb{C}$ we choose the extension given by the exponential function. We can then define (omitting E from the notation)

$$(3.26) \quad \widehat{F} = \left\{ (u, v) \in E \times E \mid \pi(u) + \pi(v) = 1 \right\}.$$

One can define log-points in $X_\alpha(F)$ as lifts of points with values in E exactly as in Definition 3.6. For any $\alpha \in \mathbb{Z}[\widetilde{S}_1 \times \widetilde{S}_1]$ a log-point \tilde{p} determines realizations $\tilde{p}(\alpha) \in \mathbb{Z}[\widehat{F}]$ and $p(\alpha) \in \mathbb{Z}[P_F^1]$ with $r(\tilde{p}(\alpha)) = p(\alpha)$ as in (3.10). Here $r: \widehat{F} \rightarrow F \setminus \{0, 1\}$ takes (u, v) to $\pi(u)$.

3.7.1. The other variant. If F does not have characteristic 2 (equivalently if $-1 \neq 1$ in F), we may also define \widehat{F}_\pm as above, but with $\epsilon_1\pi(u) + \epsilon_2\pi(v) = 1$ and $r: \widehat{F}_\pm \rightarrow F \setminus \{0, 1\}$ taking (u, v) to $\epsilon_1\pi(u)$. A log-point in $X_{\alpha, \nu}(F)$ then gives rise to a realization of α in $\mathbb{Z}[\widehat{F}_\pm]$.

4. THE LIFTED BLOCH COMPLEXES

We now define a complex $\widehat{\Gamma}(F, n)$ lifting Goncharov's complex $\Gamma(F, n)$.

4.1. Review of Goncharov's construction of $\Gamma(F, n)$. Goncharov's complex is defined in terms of groups $\mathcal{B}_k(F) = \mathbb{Z}[P_F^1]/R_k(F)$, where the $R_k(F) \subset \mathbb{Z}[P_F^1]$ are defined inductively starting with $k = 2$. For a field K , let $\mathcal{A}_2(K)$ denote the kernel of the map $\delta: \mathbb{Z}[P_K^1] \rightarrow \wedge^2(K^*)$ taking $[z]$ to $z \wedge (1 - z)$ (and $[0]$, $[1]$ and $[\infty]$ to 0). Then $R_2(F)$ is generated by $[0]$, $[\infty]$, and elements of the form $p(\alpha) - q(\alpha)$, with $\alpha \in \mathcal{A}_2(F(Y))$, and p and q are points on a geometrically irreducible smooth curve Y over F with function field $F(Y)$. If $R_{k-1}(K)$ has been defined for all fields K , there is a map $\delta: \mathbb{Z}[P_K^1] \rightarrow \mathcal{B}_{k-1}(K) \otimes K^*$ taking $[z]$ to $[z] \otimes z$ (and $[0]$ and $[\infty]$ to 0). Letting $\mathcal{A}_k(K)$ denote its kernel, $R_k(F)$ is defined as above, but with $\mathcal{A}_k(F(Y))$ instead of $\mathcal{A}_2(F(Y))$. One then shows that the δ maps induce maps

$$(4.1) \quad \delta: \mathcal{B}_k(F) \rightarrow \mathcal{B}_{k-1}(F) \otimes F^*, \quad k > 2, \quad \delta: \mathcal{B}_2(F) \rightarrow \wedge^2(F^*),$$

which induce boundary maps $\delta: \mathcal{B}_k(F) \otimes \wedge^l(F^*) \rightarrow \mathcal{B}_{k-1}(F) \otimes \wedge^{l+1}(F^*)$, taking $[z] \otimes a$ to $[z] \otimes z \wedge a$ for $k > 2$, and $\mathcal{B}_2(F) \otimes \wedge^l(F^*) \rightarrow \wedge^{l+2}(F^*)$ taking $[z] \otimes a$ to $z \wedge (1 - z) \wedge a$. This completes the construction.

Remark 4.1. Goncharov implicitly assumes that $x \wedge (-x) = x \wedge x = 0 \in \wedge^2(F^*)$ and that $[1] = 0 \in \mathcal{B}_2(F)$. It then follows that $[x] + [x^{-1}]$ and $[x] + [1 - x]$ are in $R_2(F)$ for any $x \in F$.

4.2. Overview of our construction and main results. In lifting Goncharov's construction we face two obstacles: Firstly, Goncharov allows 0, 1 and ∞ which have no lifts, and secondly, $\widehat{\mathcal{L}}_n(\tilde{p}(\alpha))$ depends on the log-branches in a seemingly non-algebraic way. The first obstacle is addressed by considering a notion of *permissible lifts of zero-degenerate points* (Definition 4.10). The second obstacle is addressed by introducing the notion of log-points *killing lower levels* (Definition 4.18), an inductive definition inspired by Proposition 3.2. It turns out that if \tilde{p} kills the lower levels of α , then $\widehat{\mathcal{L}}_n(\tilde{p}(\alpha))$ is independent of the logarithms modulo $\frac{(2\pi i)^n}{n!}$, and in order to be well defined modulo $\frac{(2\pi i)^n}{(n-1)!}$ we need the additional concept of *proper ambiguity* (Definition 4.19), a non-inductive purely symbolic property. All the above concepts may be defined over an arbitrary field.

We define $\widehat{\mathcal{B}}_n(\widehat{F}) = \mathbb{Z}[\widehat{F}]/\widehat{R}_n(F)$, where $\widehat{R}_n(F)$ is defined as follows.

Definition 4.2. The set $\tilde{R}_n(F)$ is the subset of $\mathbb{Z}[\widehat{F}]$ generated by the following two types of relations, where α denotes a differential $\widehat{\mathcal{L}}_n$ relation with proper ambiguity.

- (1) $\tilde{p}(\alpha) - \tilde{p}_0(\alpha)$, where $\tilde{p}_0(\alpha)$ kills the lower levels of α , and \tilde{p} is a lift of a point in the same geometric component of p_0 in $X_\alpha(F)$.
- (2) $\tilde{p}(\alpha)$, where the geometric component of p in $\overline{X_\alpha(F)}$ contains a zero-degenerate point with a permissible lift.

Remark 4.3. Goncharov does not need (2) since if $p_0(\alpha) \in \mathbb{Z}[\{0\}]$ for $\alpha \in \mathcal{A}_n(F(Y))$, then $p(\alpha) \in R_n(F)$. This is because $[0] \in R_n(F)$ by definition.

There are homomorphisms

$$(4.2) \quad \begin{aligned} \delta: \mathbb{Z}[\widehat{F}] &\rightarrow \widehat{\mathcal{B}}_{n-1}(\widehat{F}) \otimes E, & [(u, v)] &\mapsto [(u, v)] \otimes u, & n > 2 \\ \delta: \mathbb{Z}[\widehat{F}] &\rightarrow \wedge^2(E), & [(u, v)] &\mapsto u \wedge v. \end{aligned}$$

The proofs of the two next theorems are purely formal.

Theorem 4.4 (Proof in Section 4.5). *The projection $\mathbb{Z}[\widehat{F}] \rightarrow \mathbb{Z}[F \setminus \{0, 1\}]$ induced by $\pi: E \rightarrow F^*$ takes $\tilde{R}_n(F)$ to $R_n(F)$.*

Theorem 4.5 (Proof in Section 4.5). *The δ map takes $\tilde{R}_n(F)$ to 0 and thus descend to homomorphisms*

$$(4.3) \quad \delta: \widehat{\mathcal{B}}_n(\widehat{F}) \rightarrow \widehat{\mathcal{B}}_{n-1}(\widehat{F}) \otimes E, \quad \delta: \widehat{\mathcal{B}}_2(\widehat{F}) \rightarrow \wedge^2(E).$$

for $n > 2$ and $n = 2$, respectively.

By Theorem 4.5 we have a chain complex $\widehat{\Gamma}(F, n)$:

$$(4.4) \quad \widehat{\mathcal{B}}_n(\widehat{F}) \xrightarrow{\delta_1} \dots \xrightarrow{\delta_k} \widehat{\mathcal{B}}_{n-k}(\widehat{F}) \otimes \wedge^k(E) \xrightarrow{\delta_{k+1}} \dots \xrightarrow{\delta_{n-2}} \widehat{\mathcal{B}}_2(\widehat{F}) \otimes \wedge^{n-2}(E) \xrightarrow{\delta_{n-1}} \wedge^n(E),$$

with maps given by

$$(4.5) \quad \begin{aligned} \delta_1([(u, v)]) &= [(u, v)] \otimes u, & \delta_{n-1}([(u, v)] \otimes a) &= u \wedge v \wedge a, \\ \delta_k([(u, v)] \otimes a) &= [(u, v)] \otimes u \wedge a & \text{for } 1 < k < n-1. \end{aligned}$$

Moreover, the map

$$(4.6) \quad \widehat{\mathcal{B}}_{n-k}(\widehat{F}) \otimes \wedge^k(E) \rightarrow \mathcal{B}_{n-k}(\widehat{F}) \otimes \wedge^k(F^*), \quad [(u, v)] \otimes a \rightarrow [r(u, v)] \otimes \pi_*(a)$$

gives rise to a chain map $r: \widehat{\Gamma}(F, n) \rightarrow \Gamma(F, n)$.

The proofs of the next two theorems are significantly more involved.

Theorem 4.6 (Proof in Section 5). *If β is in $\tilde{R}_n(\mathbb{C})$ then $\widehat{\mathcal{L}}_n(\beta) = 0$ in $\mathbb{C}/\frac{(2\pi i)^n}{(n-1)!}\mathbb{Z}$.*

Theorem 4.7 (Proof in Section 6). *If $\beta \in H^1(\widehat{\Gamma}(F, n)) = \text{Ker}(\delta_1)$ we have*

$$(4.7) \quad \mathfrak{R}_n(\widehat{\mathcal{L}}_n(\beta)) = \mathcal{L}_n(r(\beta)).$$

Conjecture 4.8. $\tilde{R}_2(F)$ is generated by realizations of (inverted) lifted five term relations. $\tilde{R}_3(F)$ is generated by realizations of the 31 term relation.

Proposition 4.9. Assuming Conjecture 4.8 for $\tilde{R}_2(F)$ we have an isomorphism

$$(4.8) \quad H^1(\widehat{\Gamma}(F, 2)) \cong \widehat{\mathcal{B}}(F)$$

induced by $[(u, v)] \mapsto -[(v, u)]$. When $F = \mathbb{C}$, $\widehat{\mathcal{L}}_2$ agrees with Neumann's R .

Proof. The conjecture implies that the map induces an isomorphism between $\widehat{\mathcal{B}}_2(F)$ and $\widehat{\mathcal{P}}_E(F)$ (see Section 1.3.1). This proves the result. \square

4.3. Zero-degenerate points and permissible lifts. Let $\alpha \in \mathbb{Z}[\widetilde{\mathcal{S}}_1 \times \widetilde{\mathcal{S}}_1]$. Let \overline{I}_α be the ideal in $\mathbb{Z}[a_1, \dots, a_N]$ obtained from I_α by clearing denominators in (3.9), and let $\overline{X}_\alpha(F) = \text{Spec}(\mathbb{Z}[a_1, \dots, a_N]/\overline{I}_\alpha)$. For $q \in X_\alpha(F)$ we can still define $q(\alpha) \in \mathbb{Z}[P_F^1]$. If we introduce a formal variable $\log(0)$ we can also define a lift of q with values in $E \oplus \mathbb{Z}[\log(0)]$.

Definition 4.10. A point q in $X_\alpha(F)$ is *permissible* if either $q(z_i)$ or $q(w_i)$ is non-zero for all i . A permissible point is *zero-degenerate* if $q(\alpha) \in \mathbb{Z}[\{0\}]$. A lift \tilde{q} of a zero-degenerate point q is *permissible* if $\tilde{q}(v_i) = 0$ when $z_i = 0$ and $\tilde{q}(u_i) = 0$ when $w_i = 0$.

Example 4.11. Consider the element β from Example 3.15. For any $a, b \in F \setminus \{0, 1\}$, the point $(a_1, a_2, a_3, a_4, a_5) = (a, a, b, b, 0)$ is zero-degenerate and $(\tilde{a}, \tilde{a}, \tilde{b}, \tilde{b}, \log(0))$ is a permissible lift whenever \tilde{a} and \tilde{b} in E are lifts of a , respectively, b .

Example 4.12. In Example 3.16 a point with $a_i = 0$, $b_J = 1$ in $\overline{X}_\alpha(F)$ is zero-degenerate with a permissible lift $\tilde{a}_i = \log(0)$, $\tilde{b}_J = 0$.

4.4. Killing lower levels and proper ambiguity. We start by defining the concepts for $n = 2$. Consider the homomorphism

$$(4.9) \quad \delta: \mathbb{Z}[\widetilde{\mathcal{S}}_1 \times \widetilde{\mathcal{S}}_1] \rightarrow \wedge^2(\widetilde{\mathcal{S}}_1), \quad (u, v) \mapsto u \wedge v.$$

The following is elementary.

Lemma 4.13. An element $\alpha \in \mathbb{Z}[\widetilde{\mathcal{S}}_1 \times \widetilde{\mathcal{S}}_1]$ is a differential $\widehat{\mathcal{L}}_2$ relation if and only if $2\delta(\alpha) = 0$. If so, $\delta(\alpha) = 0$ if and only if $\sum_{i=1}^M k_{j_i} l_{j_i}$ is even for all $j = 1, \dots, N$.

Definition 4.14. A differential $\widehat{\mathcal{L}}_2$ relation α has *proper ambiguity* if $\sum_{i=1}^M k_{j_i} l_{j_i}$ is even for all $j = 1, \dots, N$.

Remark 4.15. The definition was inspired by Neumann's parity condition [Neu04, Def. 4.3], the absence of which changes his map R by 2-torsion.

Example 4.16. The five term relations (Examples 3.14 and 3.15) have proper ambiguity.

When $n = 2$ there are no lower levels, so all log-points kill the lower levels by default, and we can define $\widetilde{R}_2(F)$ as in Definition 4.2. When $n > 2$ we first define the notion of *lower level projections*.

Definition 4.17. Let $l \in 2, \dots, n-1$ and $J = \{j_1, \dots, j_{n-l}\}$ be a multisubset of $\{1, \dots, N\}$. The elements

$$(4.10) \quad \pi_J(\alpha) = \sum_{i=1}^M r_i k_{j_1 i} \cdots k_{j_{n-l} i} (u_i, v_i) \in \mathbb{Z}[\widetilde{\mathcal{S}}_1 \times \widetilde{\mathcal{S}}_1]$$

are called *level l projections* of α . We shall occasionally allow l to be n and define $\pi_\emptyset(\alpha) = \alpha$.

Definition 4.18. Let $n > 2$ and let α be a differential $\widehat{\mathcal{L}}_n$ relation. A log-point \tilde{p} *kills the lower levels* of α if

$$(4.11) \quad \tilde{p}(\pi_J(\alpha)) \in \widetilde{R}_l(F)$$

for all l and all J with $|J| = n - l$.

Definition 4.19. Let $n > 2$. A differential $\widehat{\mathcal{L}}_n$ relation α has *proper ambiguity* if

- (1) The element $\sum_{i=1}^M r_i(u_i \wedge v_i) \otimes u_i^{\otimes(n-2)}$ is zero in $\wedge^2(\tilde{S}_1) \otimes \tilde{S}_{n-2}$.
- (2) For all $j = 1, \dots, N$ the integer $\sum_{i=1}^M k_{ij}^{n-1} l_{ij}$ is divisible by n .

Note that the first condition is always satisfied up to 2-torsion (see Proposition 3.10). When $n = 2$ the two conditions coincide. We can now define $\tilde{R}_n(F)$ as in Definition 4.2.

Remark 4.20. If one excludes the proper ambiguity condition from Definition 4.2, Theorems 4.4 and 4.5 still hold modulo 2-torsion, and Theorem 4.6 holds modulo $(2\pi i)^n/n!$. Theorem 4.7 still holds.

Example 4.21. Goncharov's 22 term relation satisfies the second condition (all integers are either 0 or 3), but not the first.

Example 4.22. The 31 term relation has proper ambiguity.

Example 4.23. Each realization of an (inverted) five term relation is in $\tilde{R}_2(F)$ by Example 4.11. A straightforward (but tedious) calculation shows that all realizations of the lower level projections of the 31 term relation are linear combinations of lifted (inverted) 5 term relations. It thus follows from Example 4.12 that all realizations of the 31 term relation are in $\tilde{R}_3(F)$.

4.5. **Proof of Theorems 4.4 and 4.5.** We begin with the case $n = 2$.

Lemma 4.24. Let α be a differential $\hat{\mathcal{L}}_2$ relation with proper ambiguity. For any smooth curve Y in $X_\alpha(F)$, the element $p_Y(\alpha)$ is in $\mathcal{A}_2(F(Y))$.

Proof. The homomorphism $\pi: \tilde{S}_1 \rightarrow U$ induces a homomorphism $\pi_*: \wedge^2(\tilde{S}_1) \rightarrow \wedge^2(U)$. Since each point $p \in Y$ restricts to a homomorphism $U \rightarrow F^*$, we have a homomorphism $p_Y: U \rightarrow F(Y)^*$. We now have

$$(4.12) \quad \delta(p_Y(\alpha)) = \sum_{i=1}^M r_i(\pi(u_i)_Y) \wedge (1 - \pi(u_i)_Y) = \sum_{i=1}^M r_i \pi(u_i)_Y \wedge \pi(v_i)_Y = p_{Y^*} \circ \pi_*(\delta(\alpha)).$$

By Lemma 4.13, $\delta(\alpha) = 0$. This proves the result. \square

Lemma 4.25. The covering map r takes $\tilde{R}_2(F)$ to $R_2(F)$.

Proof. Recall that $\tilde{R}_2(F)$ is generated by the two types in Definition 4.2. Let α be a differential $\hat{\mathcal{L}}_2$ relation with proper ambiguity and let \tilde{p} and \tilde{q} be log-points in the same geometric component of $X_\alpha(F)$. Pick a geometrically irreducible curve Y containing p and q (such exists e.g. by [CP16, Cor. 1.9]). By Lemma 4.24, $p_Y(\alpha) \in \mathcal{A}_2(F(Y))$ and it follows that

$$(4.13) \quad r(\tilde{p}(\alpha) - \tilde{q}(\alpha)) = p(\alpha) - q(\alpha) \in R_2(F).$$

Similarly, if a curve Y in $X_{K,L}(F)$ containing p contains a zero-degenerate point $q \in \overline{X_\alpha(F)}$, we have $r(\tilde{p}) = p(\alpha) \in R_2(F)$. \square

Now suppose $n > 2$. Assume by induction that r maps $\tilde{R}_k(F)$ to $R_k(F)$ for all $k < n$. We begin with an elementary lemma, which holds for any integer $m \geq 2$.

Lemma 4.26. Let $\beta \in \mathbb{Z}[\tilde{S}_1 \times \tilde{S}_1]$ and let \tilde{p} be a log-point. Suppose that for a geometrically irreducible smooth curve $Y \subset X_\alpha(F)$ one has $p_Y(\beta) \in \mathcal{A}_m(F(Y))$. If $p(\beta) \in R_m(F)$ for some point p in Y , then $p_Y(\beta) \in R_m(F(Y))$.

Proof. Suppose $p_Y(\beta) = \sum_{i=1}^M r_i[x_i]$, where $x_i \in F(Y)$. The extension of scalars X of Y to $F(Y)$ is a geometrically irreducible smooth curve in $X_\alpha(F(Y))$ containing both $q = (x_1, \dots, x_M)$ and p . Moreover, $p_X(\beta) \in \mathcal{A}_m(F(Y)(X))$. It follows that $p(\beta) - q(\beta) \in R_m(F(Y))$ and since $p(\beta) \in R_m(F) \subset R_m(F(Y))$, it follows that $q(\beta) = p_Y(\beta) \in R_m(F(Y))$. \square

Lemma 4.27. For $n > 2$, α is a differential $\widehat{\mathcal{L}}_n$ relation if and only if all level l projections of α are differential $\widehat{\mathcal{L}}_l$ relations, which again holds if and only if all level 2 projections are differential $\widehat{\mathcal{L}}_2$ relations. A differential $\widehat{\mathcal{L}}_n$ relation satisfies the first proper ambiguity condition if and only if all level 2 projections have proper ambiguity.

Proof. This follows from the fact that the multiplication map $\Omega_{l-1}^1(\widetilde{S}) \otimes \widetilde{S}_{n-l} \rightarrow \Omega_{n-1}^1(\widetilde{S})$ is an isomorphism for all l . \square

Lemma 4.28. Let α be a differential $\widehat{\mathcal{L}}_n$ relation with proper ambiguity and \widetilde{p} a log-point killing the lower levels of α . For any smooth curve Y in $X_\alpha(F)$ containing p , $p_Y(\alpha) \in \mathcal{A}_n(F(Y))$.

Proof. Our induction hypothesis that r maps $\widetilde{R}_k(F)$ to $R_k(F)$ for all $k < n$ implies in particular that $p(\pi_{J_m}(\alpha))$ is in $R_m(F)$ for all $n - m$ element multisets J_m . Let us prove by induction on m that

$$(4.14) \quad p_Y(\pi_{J_m}(\alpha)) \in \mathcal{A}_m(F(Y))$$

holds for all $m = 2, \dots, n$ and all J_m . The case $m = n$ is the desired statement, and the case $m = 2$ follows from Lemmas 4.27 and 4.24. Suppose by induction that (4.14) holds for $m = k > 2$. For any J_{k+1} we have

$$(4.15) \quad \delta(p_Y(\pi_{J_{k+1}}(\alpha))) = \sum_{j=1}^N p_Y(\pi_{J \cup \{j\}}(\alpha)) \otimes p_Y(a_j),$$

which by Lemma 4.26 is zero in $\mathcal{B}_k(F(Y)) \otimes F(Y)^*$. Hence, $p_Y(\pi_{J_{k+1}}(\alpha)) \in \mathcal{A}_{k+1}(F(Y))$. \square

Theorem 4.29. The covering map r takes $\widetilde{R}_n(F)$ maps to $R_n(F)$.

Proof. The proof is the same as that of Lemma 4.25 except that Lemma 4.28 is used instead of Lemma 4.24. \square

Theorem 4.30. The map δ from (4.2) takes $\widetilde{R}_n(F)$ to 0.

Proof. Let α be a differential $\widehat{\mathcal{L}}_n$ relation and let \widetilde{p} be a log-point killing the lower levels of α . The lift \widetilde{p} induces a homomorphism $\widetilde{p}_*: \wedge^2(\widetilde{S}_1) \rightarrow \wedge^2(E)$, and the result for $n = 2$ now follows from the fact that $\widetilde{p}_*(\delta(\alpha)) = \delta(\widetilde{p}(\alpha))$. For $n > 2$, we have

$$(4.16) \quad \delta(\widetilde{p}(\alpha)) = \sum_{i=1}^K r_i[(\widetilde{p}(u_i), \widetilde{p}(v_i))] \otimes \widetilde{p}(u_i) = \sum_{j=1}^N \sum_{i=1}^K r_i k_{ji}[(\widetilde{p}(u_i), \widetilde{p}(v_i))] \otimes \widetilde{p}(a_j).$$

Since \widetilde{p} kills the lower levels of α , $\sum_{i=1}^K r_i k_{ji}[(\widetilde{p}(u_i), \widetilde{p}(v_i))] \in \widetilde{R}_{n-1}(F)$ for all j , and the result follows. \square

4.6. A variant using \widehat{F}_\pm . Now assume that $-1 \neq 1 \in F$. By analogy with the case $F = \mathbb{C}$, we denote the unique element $e \in E$ with $2e = \iota(1)$ by πi . Since realizations in \widehat{F}_\pm depend on a choice of sign determination, we need a notion of when two realizations are equivalent.

Definition 4.31. Let \mathcal{V} and \mathcal{V}' be sign determinations and let \tilde{p} and \tilde{q} be log-points in $X_{K,L,\mathcal{V}}(F)$ and $X_{K,L,\mathcal{V}'}(F)$, respectively. We say that \tilde{p} and \tilde{q} are *sign equivalent* if there is a point $r \in X_{K,L,\mathcal{V}}$ in the same geometric component as p and a homomorphism $\phi: \tilde{S}_1 \rightarrow \mathbb{Z}$ such that $\tilde{q} + \pi i \phi$ is a lift of r .

Example 4.32. $[(\text{Log}(x), \text{Log}(1-x))]$ and $[(\text{Log}(-y) - 2\pi i, \text{Log}(1-y) + 2\pi i)]$ are sign equivalent realizations of $[(\tilde{a}_1, \tilde{a}_2)]$ in $\mathbb{Z}[\widehat{\mathbb{C}}_{\pm}]$.

The notion of permissible lifts of zero-degenerate points is the same for points in $\overline{X_{\alpha,\mathcal{V}}(F)}$ and the notion of proper ambiguity is unchanged. The notion of killing lower levels now involves $\tilde{R}_l(F)_{\pm}$. We can thus define $\tilde{R}_n(F)_{\pm}$ as in Definition 4.2, but with $p_0 \in X_{\alpha,\mathcal{V}}(F)$ and \tilde{p} sign equivalent to \tilde{p}_0 . We can now define groups $\widehat{\mathcal{B}}_k(F)_{\pm} = \mathbb{Z}[\widehat{F}_{\pm}]/R_k(F)_{\pm}$, which fit in a chain complex $\widehat{\Gamma}(F, n)_{\pm}$. The only thing that changes is that only $2\tilde{R}_n(F)_{\pm}$ maps to $R_n(F)$, and that $\widehat{\mathcal{L}}_n$ only takes $\tilde{R}_n(\mathbb{C})_{\pm}$ to 0 modulo $\frac{(\pi i)^n}{(n-1)!}$. All proofs are identical to the case of \widehat{F} (see Remark 5.3).

Remark 4.33. One can show that if $\alpha \in \mathbb{Z}[\tilde{S}_1 \times \tilde{S}_1]$ is Goncharov's 22 term relation, then any log-point \tilde{p} kills the lower levels of 2α . Since 2α has proper ambiguity, it follows that $2\tilde{p}(\alpha)$ is constant in $H^1(\widehat{\Gamma}(F, 3)_{\pm})$. When $F = \mathbb{C}$ it follows from Theorem 4.7 that $\widehat{\mathcal{L}}_3(\tilde{p}(\alpha)) = 3\zeta(3)$ modulo $\frac{(\pi i)^3}{4}$. With extra effort one can replace the denominator by 2.

5. PROOF OF THEOREM 4.6

We now prove that $\widehat{\mathcal{L}}_n(\beta) = 0$ if $\beta \in \tilde{R}_n(\mathbb{C})$. This is an immediate consequence of the following result together with Lemma 2.12.

Theorem 5.1. *Let α be a differential $\widehat{\mathcal{L}}_n$ relation and p a point in $X_{\alpha}(\mathbb{C})$. If some lift of p kills the lower levels of α , then $\widehat{\mathcal{L}}_n(\tilde{p}(\alpha))$ is independent of the choice of lift of p modulo $\frac{(2\pi i)^n}{n!}$. If α has proper ambiguity, then this holds modulo $\frac{(2\pi i)^n}{(n-1)!}$.*

In order to prove this result we begin with a technical lemma comparing the values of $\widehat{\mathcal{L}}_n$ at two points with the same image in $\mathbb{C} \setminus \{0, 1\}$. Although our main interest is in $\widehat{\mathbb{C}}$ we formulate it for $\widehat{\mathbb{C}}_{\pm}$. The reason for the particular way of writing the right-hand side will become clear later.

Lemma 5.2. Let $\bar{k} = 2\pi i k$ and $\bar{l} = 2\pi i l$ with $k, l \in \frac{1}{2}\mathbb{Z}$. We have

$$(5.1) \quad \widehat{\mathcal{L}}_n(u+\bar{k}, v+\bar{l}) - \widehat{\mathcal{L}}_n(u, v) = \sum_{r=1}^{n-2} (-1)^r \frac{\bar{k}^r}{r!} \widehat{\mathcal{L}}_{n-r}(u, v) + \frac{(-1)^n}{n!} \left(A(u, v; \bar{k}, \bar{l}) + \sum_{r=0}^{n-3} A_r(u, v; \bar{k}, \bar{l}) - \bar{k}^{n-1} \bar{l} \right),$$

modulo $\frac{(\pi i)^n}{(n-1)!}$ where

$$(5.2) \quad A(u, v; \bar{k}, \bar{l}) = (\bar{k}v - \bar{l}u)(u + \bar{k})^{n-2}, \quad A_r(u, v; \bar{k}, \bar{l}) = (\bar{k}v - \bar{l}u) \binom{n-2}{r+1} u^r \bar{k}^{n-2-r}.$$

Moreover, if $(u, v) \in \widehat{\mathbb{C}}_{++}$ and $k, l \in \mathbb{Z}$, then (5.1) holds modulo $\frac{(2\pi i)^n}{(n-1)!}$.

Proof. Let's define functions X , Y and Z of $(u, v) \in \widehat{\mathbb{C}}_{\pm}$ as follows:

$$(5.3) \quad X = \frac{n!}{(-1)^n} (\widehat{\mathcal{L}}_n(u + \bar{k}, v + \bar{l}) - \widehat{\mathcal{L}}_n(u, v)), \quad Y = \frac{n!}{(-1)^n} \left(\sum_{r=1}^{n-2} \frac{(-1)^r}{r!} \bar{k}^r \widehat{\mathcal{L}}_{n-r}(u, v) \right)$$

$$Z = A(u, v; \bar{k}, \bar{l}) + \sum_{r=0}^{n-3} A_r(u, v; \bar{k}, \bar{l}) - \bar{k}^{n-1} \bar{l}$$

We must show that $X - Y = Z$. We first show that $dX - dY = dZ$. Using the fact that $d\widehat{\mathcal{L}}_k = \omega_k$, we obtain

$$(5.4) \quad dX = (n-1) \left((u + \bar{k})^{n-2} ((u + \bar{k})dv - (v + \bar{l})du) - u^{n-2}(udv - vdu) \right)$$

$$dY = \sum_{r=1}^{n-2} \bar{k}^r \binom{n}{r} (n-r-1) u^{r-2} (udv - vdu).$$

It follows that $dX - dY$ equals

$$(5.5) \quad \sum_{r=1}^{n-2} \left(((n-1) \binom{n-2}{r} - \binom{n}{r} (n-r-1)) u^{n-r-2} \bar{k}^r (udv - vdu) + (n-1) (u + \bar{k})^{n-2} (\bar{k}dv - \bar{l}du) \right).$$

One now shows that $dX - dY = dZ$ by a term by term comparison. The coefficient of dv in dZ equals

$$(5.6) \quad \bar{k} (u + \bar{k})^{n-2} + \sum_{r=0}^{n-3} \bar{k} \binom{n-2}{r+1} u^r \bar{k}^{n-2-r}.$$

The coefficient of $u^s dv$ in $dX - dY$ is $n-1$ when $s=0$ and

$$(5.7) \quad (n-1) \binom{n-2}{n-1-s} - s \binom{n}{n-1-s} + (n-1) \binom{n-2}{s} = \binom{n-2}{s} + \binom{n-2}{s+1},$$

when $s \in \{1, \dots, n-1\}$. By (5.6) this agrees with the coefficient of $u^s dv$ in dZ . A similar consideration comparing coefficients of du completes the proof that $dX - dY = dZ$.

One now need only show that $X - Y - Z = 0$ for a single point in each of the 4 components of $\widehat{\mathbb{C}}_{\pm}$. We choose points

$$(5.8) \quad (\pi i, \log(2)), \quad (0, \log(2)), \quad (0, \log(2) + \pi i), \quad (\pi i, \log(2) + \pi i)$$

in $\widehat{\mathbb{C}}_{++}$, $\widehat{\mathbb{C}}_{-+}$, $\widehat{\mathbb{C}}_{--}$, $\widehat{\mathbb{C}}_{+-}$, respectively. For $(u, v) = (\pi i, \log(2))$ we have

$$(5.9) \quad X = \frac{n!}{(-1)^n} \left(\sum_{r=0}^{n-1} \frac{(-1)^r}{r!} \text{Li}_{n-r}(-1) ((\pi i + \bar{k})^r - (\pi i)^r) \right) - (\pi i + \bar{k})^{n-1} (\log(2) + \bar{l}) + (\pi i)^{n-1} \log(2)$$

and

$$(5.10) \quad Y = \frac{n!}{(-1)^n} \sum_{t=1}^{n-2} \frac{(-1)^t}{t!} \bar{k}^t \left(\sum_{s=0}^{n-t-1} \frac{(-1)^s}{s!} \text{Li}_{n-t-s}(-1) (\pi i)^s - \frac{(-1)^{n-t}}{(n-t)!} (\pi i)^{n-t-1} \log(2) \right)$$

$$= n \bar{k}^{n-1} \text{Li}_1(-1) + \frac{n!}{(-1)^n} \left(\sum_{r=1}^{n-1} ((\pi i + \bar{k})^r - (\pi i)^r) \frac{(-1)^r}{r!} \text{Li}_{n-r}(-1) \right) +$$

$$- \sum_{t=1}^{n-2} \binom{n}{t} \bar{k}^t (\pi i)^{n-t-1} \log(2).$$

Hence,

(5.11)

$$\begin{aligned} X - Y &= (\pi i)^{n-1} \log(2) - (\pi i + \bar{k})^{n-1} (\log(2) + \bar{l}) + \sum_{t=1}^{n-2} \binom{n}{t} \bar{k}^t (\pi i)^{n-t-1} \log(2) - n \bar{k}^{n-1} \text{Li}_1(-1) \\ &= \log(2) \left((\pi i)^{n-1} - (\pi i + \bar{k})^{n-1} + \sum_{r=0}^{n-3} \binom{n}{r+2} \bar{k}^{n-2-r} (\pi i)^{r+1} + n \bar{k}^{n-1} \right) - \bar{l} (\pi i + \bar{k})^{n-1}. \end{aligned}$$

Similarly, we obtain

$$(5.12) \quad Z = \log(2) \left(\bar{k} (\pi i + \bar{k})^{n-2} + \sum_{r=0}^{n-3} \bar{k} \binom{n-2}{r+1} (\pi i)^r \bar{k}^{n-2-r} \right) - \bar{l} (\pi i + \bar{k})^{n-1}.$$

Letting

$$(5.13) \quad a_r = \pi i \binom{n-2}{r+2} \bar{k}^{n-2-r} (\pi i)^r, \quad b_r = \pi i \binom{n-2}{r+1} \bar{k}^{n-2-r} (\pi i)^r,$$

(5.11) and (5.12) together with the equality $\binom{n}{r+2} = \binom{n-2}{r+2} + 2\binom{n-2}{r+1} + \binom{n-2}{r}$ imply that we have

$$(5.14) \quad \begin{aligned} X - Y - Z &= \log(2) \left(n \bar{k}^{n-1} + \sum_{r=0}^{n-2} (a_r - a_{r-1}) + 2 \sum_{r=0}^{n-2} (b_r - b_{r-1}) \right) \\ &= \log(2) \left(n \bar{k}^{n-1} - a_{-1} - 2b_{-1} \right) = 0. \end{aligned}$$

This concludes the proof when $(u, v) = (\pi i, \log(2)) \in \mathbb{C}_{++}$. The computations for the other 3 points are similar (and much simpler for $(0, \log(2))$ for $(0, \log(2) + \pi i)$). \square

Proof of Theorem 5.1. Now suppose α is a differential $\widehat{\mathcal{L}}_n$ relation with proper ambiguity and that \tilde{p} kills the lower levels of α . For $j \in \{1, \dots, N\}$ let $T_j(\tilde{p})$ be the lift obtained from \tilde{p} by adding $2\pi i$ to \tilde{a}_j . It is enough to prove that $\widehat{\mathcal{L}}_n(T_j(\tilde{p})(\alpha)) = \widehat{\mathcal{L}}_n(\tilde{p}(\alpha))$ for all j . For an integer k , let $\bar{k} = 2\pi i k$. By Lemma 5.2, we have

$$(5.15) \quad \begin{aligned} \widehat{\mathcal{L}}_n(T_j(\tilde{p})(\alpha)) - \widehat{\mathcal{L}}_n(\tilde{p}(\alpha)) &= \sum_{i=1}^M r_i \left(\sum_{r=1}^{n-2} (-1)^r \frac{\bar{k}^r}{r!} \widehat{\mathcal{L}}_{n-r}(\tilde{p}(u_i), \tilde{p}(v_i)) \right) + \\ &\frac{(-1)^n}{n!} \sum_{i=1}^M r_i A(\tilde{p}(u_i), \tilde{p}(v_i); \bar{k}_{ji}, \bar{l}_{ji}) + \frac{(-1)^n}{n!} \sum_{i=1}^M r_i \left(\sum_{r=0}^{n-3} A_r(\tilde{p}(u_i), \tilde{p}(v_i); \bar{k}_{ji}, \bar{l}_{ji}) - \right. \\ &\left. \frac{(-1)^n}{n!} \sum_{i=1}^M r_i (\bar{k}_{ij})^{n-1} (\bar{l}_{ij}). \right) \end{aligned}$$

The first sum vanishes since \tilde{p} kills the lower levels of α (consider $J = \{j, \dots, j\}$). As we shall see, the second and third sum vanish since α is a differential $\widehat{\mathcal{L}}_n$ relation, and the last term vanishes by the (second) proper ambiguity condition. To see this consider the polynomial ring $\widetilde{S}[\tilde{a}_{2\pi i}]$ obtained from \widetilde{S} by adjoining a variable $\tilde{a}_{2\pi i}$, which we think of as a symbolic representation of $2\pi i \in \mathbb{C}$. The homomorphism $\tilde{p}: \widetilde{S}_1 \rightarrow \mathbb{C}$ extends canonically to $\widetilde{S}[\tilde{a}_{2\pi i}]_1$ by taking $\tilde{a}_{2\pi i}$ to $2\pi i$. Note that $T_j(\tilde{p})(\alpha) = \tilde{p}(T_j(\alpha))$, where $T_j: \widetilde{S}_1 \rightarrow \widetilde{S}[\tilde{a}_{2\pi i}]_1$ is the homomorphism taking \tilde{a}_j to

$\tilde{a}_j + \tilde{a}_{2\pi i}$ and fixing all other generators. We have homomorphisms

$$(5.16) \quad \begin{aligned} \chi: \wedge^2(\tilde{S}[\tilde{a}_{2\pi i}]_1) &\rightarrow \tilde{S}_1, & (u + p\tilde{a}_{2\pi i}) \wedge (v + q\tilde{a}_{2\pi i}) &\mapsto pv - qu \\ \wedge^2 \text{Sym}^{n-2}: \mathbb{Z}[\tilde{S}_1 \times \tilde{S}_1] &\rightarrow \wedge^2(\tilde{S}_1) \otimes \tilde{S}_{n-2}, & (u, v) &\mapsto (u \wedge v) \otimes u^{n-2} \end{aligned}$$

as well as projection homomorphisms

$$(5.17) \quad \Pi_k: \tilde{S}[\tilde{a}_{2\pi i}]_{n-2} \rightarrow \tilde{S}_k$$

defined by taking a monomial x to x/\tilde{a}_π^{n-2-k} if x is divisible by \tilde{a}_π $n-2-k$ times and 0 otherwise.

Letting $m: \tilde{S}[\tilde{a}_{2\pi i}]_k \otimes \tilde{S}[\tilde{a}_{2\pi i}]_l \rightarrow \tilde{S}[\tilde{a}_{2\pi i}]_{k+l}$ be the multiplication map, the definition of the maps imply that

$$(5.18) \quad m \circ (\chi \otimes \text{id}) \circ T_{j*}(\wedge^2 \text{Sym}^{n-2}(\alpha)) = \sum_{i=1}^K r_i (k_{ji}v_i - l_{ji}u_i)(u_i + k_{ji}\tilde{a}_\pi)^{n-2} \in \tilde{S}[\tilde{a}_{2\pi i}]_{n-1}.$$

By multiplication, \tilde{p} induces a homomorphism $\tilde{p}: \tilde{S}[\tilde{a}_{2\pi i}]_{n-1} \rightarrow \mathbb{C}$, which takes the right-hand side of (5.18) to $\sum_{i=1}^M r_i A(\tilde{p}(u_i), \tilde{p}(v_i); \bar{k}_{ji}, \bar{l}_{ji})$. Since α is a differential $\widehat{\mathcal{L}}_n$ relation, the left-hand side of (5.18) is 0 since $\wedge^2 \text{Sym}^{n-2}(2\alpha) = 0$. This proves the vanishing of the second sum. The vanishing of the third sum is proved similarly, using that

$$(5.19) \quad m \circ (\chi \otimes \text{id}) \circ T_{j*}(\wedge^2 \text{Sym}^{n-2}(\alpha)) = \sum_{i=1}^K r_i (k_{ji}v_i - l_{ji}u_i)(u_i + k_{ji}\tilde{a}_\pi)^{n-2} \in \tilde{S}[\pi]_{n-1},$$

which holds for any r . Finally, the vanishing of the last sum follows from the second proper ambiguity condition. This concludes the proof of Theorem 5.1. \square

Remark 5.3. The same argument shows that $\widehat{\mathcal{L}}_n(\beta) = 0$ modulo $\frac{(\pi i)^n}{(n-1)!}$ if β is in the subgroup $\tilde{R}(\mathbb{C})_\pm$ defined in Section 4.6. The only difference is that adding a half integral multiple of $2\pi i$ to \tilde{a}_j changes the lift by a sign equivalence.

6. COMPARING $\widehat{\mathcal{L}}_n$ AND \mathcal{L}_n

We now prove Theorem 2.10 and Theorem 4.7. Recall that

$$(6.1) \quad \mathcal{L}_n(z) = \Re_n \left(\sum_{r=0}^{n-1} \beta_r \text{Li}_{n-r}(z) \text{Log}(|z|) \right), \quad \beta_r = \frac{2^r}{r!} B_r.$$

Define signs

$$(6.2) \quad \eta_j = \begin{cases} (-1)^{\frac{j(j-1)}{2}} & n \text{ even} \\ (-1)^{\frac{j(j+1)}{2}} & n \text{ odd} \end{cases}, \quad \epsilon_j = \begin{cases} (-1)^{\frac{j}{2}} & j \text{ even} \\ (-1)^{\frac{j+1}{2}} & j \text{ odd}. \end{cases}$$

Let $\mathbf{1}_{\text{even}}$ and $\mathbf{1}_{\text{odd}}$ denote the characteristic functions for the even and odd numbers respectively. For non-negative integers i and j let

$$(6.3) \quad c_i = (1 - 2^{1-i})\beta_i, \quad c_{i,j} = \frac{c_i}{j!} \eta_j, \quad d_{i,j} = (-1)^{\frac{i+j}{2}} \epsilon_n \sum_{r=0}^i \frac{c_r}{(i+j+2-r)!} \mathbf{1}_{\text{even}}(i).$$

In particular, $c_0 = -1$. Note that up to a sign, the $c_{i,j}$ and $d_{i,j}$ are independent of n , and 0 when i is odd. We wish to prove that

$$(6.4) \quad \mathfrak{R}_n(\widehat{\mathcal{L}}_n(u, v)) - \mathcal{L}_n(z) = \sum_{s=1}^{n-2} \left(\mathfrak{R}_{n-s}(\widehat{\mathcal{L}}_{n-s}(u, v)) \sum_{i=0}^s c_{i,s-i} \operatorname{Re}(u)^i \operatorname{Im}(u)^j \right) \\ + \det(u \wedge v) \sum_{i=0}^{n-2} d_{i,n-2-i} \operatorname{Re}(u)^i \operatorname{Im}(u)^j,$$

where $z = r(u, v) \in \mathbb{C} \setminus \{0, 1\}$. We do this directly by expanding both sides of (6.4) and comparing terms. For notational simplicity let $\operatorname{Li}_r(z) = x_r + iy_r$ and $\operatorname{Log}(z) = a + bi$. We shall only compare the terms involving x_m and leave the analogous comparison of y_m terms and terms not involving any x_m or y_m to the reader. Letting $\operatorname{Coeff}_{LHS}(x_m)$ and $\operatorname{Coeff}_{RHS}(x_m)$ denote the coefficients of x_m when expanding the lefthand, respectively, righthand side of (6.4), we thus wish to prove that $\operatorname{Coeff}_{LHS}(x_m) = \operatorname{Coeff}_{RHS}(x_m)$ for all m . We assume for notational simplicity that $(u, v) = (a + bi + 2p\pi i, -x_1 - iy_1 + 2q\pi i) \in \widehat{\mathbb{C}}_{++}$.

6.1. The lefthand side. As a simple consequence of the formula (2.6) we have

$$(6.5) \quad \widehat{\mathcal{L}}_n(u, v) = \sum_{r=0}^{n-1} \frac{1}{r!} (-1)^r x_{n-r} (a + (b + 2p\pi)i)^r + i \sum_{r=0}^{n-1} \frac{1}{r!} (-1)^r y_{n-r} (a + (b + 2p\pi)i)^r \\ - (-1)^n \frac{n-1}{n!} x_1 (a + (b + 2p\pi)i)^{n-1} - i (-1)^n \frac{n-1}{n!} y_1 (a + (b + 2p\pi)i)^{n-1} \\ - \frac{2q\pi i}{(n-1)!} (-2p\pi i)^{n-1} + \frac{2q\pi i}{n!} (-1)^{n-1} (a + (b + 2p\pi)i)^{n-1}.$$

Using (6.5) and (6.1) we see that

$$(6.6) \quad \operatorname{Coeff}_{LHS}(x_m) = \frac{(-1)^{n-m}}{(n-m)!} \mathfrak{R}_n \left((a + (b + 2p\pi)i)^{n-m} \right) - a^{n-m} \beta_{n-m} \mathbf{1}_{\text{odd}}(n) \quad \text{for } 1 < m \leq n, \\ \operatorname{Coeff}_{LHS}(x_1) = (-1)^{n-1} \frac{n-1}{n!} \mathfrak{R}_n \left((a + (b + 2p\pi)i)^{n-1} \right) - a^{n-1} \beta_{n-1} \mathbf{1}_{\text{odd}}(n).$$

Lemma 6.1. We have

$$(6.7) \quad \mathfrak{R}_s((a + (b + 2p\pi)i)^{s-m}) = \sum_{k+l=s-m} \binom{s-m}{k} (-1)^{\frac{m+k+1}{2}} \epsilon_s \mathbf{1}_{\text{odd}}(m+k) a^k (b + 2p\pi)^l.$$

Proof. This is an elementary consequence of the binomial theorem. \square

It thus follows that $\operatorname{Coeff}_{LHS}(x_m)$ can be written as sums of terms of the form $a^k (b + 2p\pi)^l$ where $k+l+m = n$. Let $\operatorname{Coeff}_{LHS}(x_m, k, l)$ denote the coefficient of $a^k (b + 2p\pi)^l$ in $\operatorname{Coeff}_{LHS}(x_m)$. By Lemma 6.1 it follows from (6.6) that

$$(6.8) \quad \operatorname{Coeff}_{LHS}(x_m, k, l) = \begin{cases} \frac{(-1)^{n-m}}{k!l!} (-1)^{\frac{m+k+1}{2}} \epsilon_n \mathbf{1}_{\text{odd}}(m+k) & \text{for } l > 0 \\ \frac{(-1)^{n-m}}{k!l!} (-1)^{\frac{m+k+1}{2}} \epsilon_n \mathbf{1}_{\text{odd}}(m+k) - \mathbf{1}_{\text{odd}}(n) \beta_k & \text{for } l = 0 \end{cases}$$

for $m > 1$ and that

$$(6.9) \quad \operatorname{Coeff}_{LHS}(x_1, k, l) \begin{cases} \frac{(-1)^{n-1}}{k!l!} \frac{n-1}{n} (-1)^{\frac{k+2}{2}} \epsilon_n \mathbf{1}_{\text{even}}(k) & \text{for } l > 0 \\ \frac{(-1)^{n-1}}{k!l!} \frac{n-1}{n} (-1)^{\frac{k+2}{2}} \epsilon_n \mathbf{1}_{\text{even}}(k) - \mathbf{1}_{\text{odd}}(n) \beta_k & \text{for } l = 0. \end{cases}$$

6.2. The righthand side. We shall need the following technical lemmas

Lemma 6.2. For any non-negative integers l and s we have

$$(6.10) \quad \sum_{j=0}^l (-1)^j \binom{l}{j} = 0 \text{ for } l > 0, \quad \sum_{j=0}^l \frac{(-1)^j \binom{l}{j}}{s+l-j} = \frac{(-1)^l}{s \binom{l+s}{l}}.$$

Proof. The first is elementary and the second can be found in [SWZ04, eq. (5)]. \square

Lemma 6.3. Let k and l be non-negative integers with $l > 1$ odd.

$$(6.11) \quad \sum_{i=0}^k \frac{c_i}{(k-i)!} = (-1)^{k-1} \beta_k, \quad \sum_{i=0}^{l-1} \frac{l-1-i}{(l-i)!} c_i = -\beta_{l-1}$$

Proof. Since $\sum_{r=0}^{\infty} \frac{B_r}{r!} x^r = \frac{x}{e^x-1}$ it follows that the generating function for β_i is the function $f(x) = \frac{2x}{e^{2x}-1}$. The first equation now follows from the fact that $e^x(f(x) - 2f(x/2)) = -f(-x)$ and the second from the fact that $(\cosh(x) - \frac{\sinh(x)}{x})(f(x) - 2f(x/2)) = 1 - x - f(x)$. \square

Since $(u, v) = (a + bi + 2p\pi i, -x_1 - iy_1 + 2q\pi i)$ it follows that

$$(6.12) \quad \det(u \wedge v) = x_1(b + 2p\pi) - y_1 a + 2\pi q a.$$

The right hand side of (6.4) expands to

$$(6.13) \quad \sum_{s=2}^{n-1} \left(\mathfrak{R}_s(\widehat{\mathcal{L}}_s) \sum_{i+j=n-s} c_{i,j} a^i (b+2p\pi)^j \right) + (x_1(b+2p\pi) - y_1 a + 2\pi q a) \sum_{i+j=n-2} d_{i,j} a^i (b+2p\pi)^j$$

and it follows from (6.5) that we have

$$(6.14) \quad \begin{aligned} \text{Coeff}_{RHS}(x_m) &= \sum_{s=m}^{n-1} \left(\frac{(-1)^{s-m}}{(s-m)!} \mathfrak{R}_s \left((a + (b+2p\pi)i)^{s-m} \right) \sum_{i+j=n-s} c_{i,j} a^i (b+2p\pi)^j \right) \\ \text{Coeff}_{RHS}(x_1) &= \sum_{s=2}^{n-1} \left((-1)^{s-1} \frac{s-1}{s!} \mathfrak{R}_s \left((a + (b+2p\pi)i)^{s-1} \right) \sum_{i+j=n-s} c_{i,j} a^i (b+2p\pi)^j \right) \\ &\quad + (b+2p\pi) \sum_{i+j=n-2} d_{i,j} a^i (b+2p\pi)^j. \end{aligned}$$

We can thus define $\text{Coeff}_{RHS}(x_m, k, l)$ for $k+l+m = n$ as above. Let's first assume that $m > 1$. By Lemma 6.1 $\text{Coeff}_{RHS}(x_m, k, l)$ is given by

$$(6.15) \quad \begin{aligned} &\sum_{i=0}^k \sum_{\substack{j=0 \\ (i,j) \neq (0,0)}}^l \frac{(-1)^{n-i-j-m}}{(n-i-j-m)!} \binom{n-i-j-m}{k-i} (-1)^{\frac{m+k-i+1}{2}} \epsilon_{n-i-j} \mathbf{1}_{\text{odd}}(m+k-i) c_{i,j} \\ &= (-1)^{n-m} (-1)^{\frac{m+k+1}{2}} \epsilon_n \mathbf{1}_{\text{odd}}(m+k) \left(\sum_{i=0}^k \sum_{j=0}^l \frac{c_i (-1)^{-j} \binom{l}{j}}{(k-i)! j!} - \frac{c_0}{k! l!} \right), \end{aligned}$$

which follows from the fact that $(-1)^{\frac{i}{2}} \epsilon_{n-i-j} = \epsilon_{n-j} = \epsilon_n \eta_j$ whenever i is even. By Lemma 6.2 and Lemma 6.3 it follows that this agrees with (6.8). We have thus proved that $\text{Coeff}_{LHS}(x_m) = \text{Coeff}_{RHS}(x_m)$ for $m > 1$.

Now let $m = 1$ and suppose $l > 0$. By (6.14) we see that $\text{Coeff}(x_1, k, l)$ equals

$$\begin{aligned}
(6.16) \quad & d_{k,l-1} + \sum_{\substack{i=0 \\ i \text{ even}}}^k \sum_{\substack{j=0 \\ (i,j) \neq (0,0)}}^l \frac{c_{i,j}(-1)^{n-j-1}(n-i-j-1)(-1)^{\frac{k+2}{2}} \epsilon_n \mathbf{1}_{\text{odd}}(1+k)}{(n-i-j)(k-i)!(l-j)!} \\
& = d_{k,l-1} + (-1)^{n-1}(-1)^{\frac{k+2}{2}} \epsilon_n \mathbf{1}_{\text{even}}(k) \left(\sum_{\substack{i=0 \\ i \text{ even}}}^k \sum_{j=0}^l \frac{c_i(-1)^{-j}(n-i-j-1) \binom{l}{j}}{(n-i-j)(k-i)!!} - \frac{c_0(n-1)}{nk!!} \right) \\
& = d_{k,l-1} + (-1)^{n-1}(-1)^{\frac{k+2}{2}} \epsilon_n \mathbf{1}_{\text{even}}(k) \left(- \sum_{\substack{i=0 \\ i \text{ even}}}^k \sum_{j=0}^l \frac{c_i(-1)^{-j} \binom{l}{j}}{(n-i-j)(k-i)!!} - \frac{c_0(n-1)}{nk!!} \right) \\
& = d_{k,l-1} - (-1)^{n-1}(-1)^{\frac{k+2}{2}} \epsilon_n \mathbf{1}_{\text{even}}(k) (-1)^l \sum_{\substack{i=0 \\ i \text{ even}}}^k \frac{c_i}{(n-i)!} + (-1)^{n-1}(-1)^{\frac{k+2}{2}} \epsilon_n \mathbf{1}_{\text{even}}(k) \\
& = (-1)^{n-1}(-1)^{\frac{k+2}{2}} \epsilon_n \mathbf{1}_{\text{even}}(k),
\end{aligned}$$

where the second last equality follows from Lemma 6.2. The fact that this equals (6.9) follows from Lemma 6.3.

Finally, when $l = 0$ (so that $k = n - 1$) a similar computation shows that $\text{Coeff}_{RHS}(x_1, k, 0)$ is given by

$$(6.17) \quad (-1)^{n-1}(-1)^{\frac{k+2}{2}} \epsilon_n \mathbf{1}_{\text{even}}(k) \left(\sum_{\substack{i=0 \\ i \text{ even}}}^{n-1} \frac{n-1-i}{(n-i)!} c_i - \frac{c_0(n-1)}{nk!!} \right).$$

The fact that this agrees with (6.9) follows from Lemma 6.3. This concludes the proof of Theorem 2.10.

6.3. Proof of Theorem 4.7. For $s = 1, \dots, n - 2$, let

$$(6.18) \quad \Psi_s : \mathbb{Z}[\widehat{\mathbb{C}}] \rightarrow \widehat{\mathcal{B}}_{n-s}(\widehat{\mathbb{C}}) \otimes \mathbb{C}^{\otimes s}, \quad [(u, v)] \mapsto [(u, v)] \otimes u^{\otimes s}.$$

We note that

$$(6.19) \quad \Psi_s = \overbrace{(\delta \otimes \text{id}) \circ \dots \circ (\delta \otimes \text{id})}^{s-1} \circ \delta.$$

Also, let (for $i = 0, \dots, s$)

$$(6.20) \quad \text{ReIm}_i : \mathbb{C}^{\otimes s} \rightarrow \mathbb{R}, \quad z_1 \otimes \dots \otimes z_s \mapsto \prod_{k=1}^i \text{Re}(z_k) \prod_{k=i+1}^s \text{Im}(z_k).$$

Letting $c_{i,j}$ and $d_{i,j}$ be as above, define

$$(6.21) \quad C_s = \sum_{i=0}^s c_{i,s-i} \text{ReIm}_i : \mathbb{C}^{\otimes s} \rightarrow \mathbb{R}, \quad D = \sum_{i=0}^{n-2} d_{i,n-2-i} \text{ReIm}_i : \mathbb{C}^{\otimes n-2} \rightarrow \mathbb{R}.$$

Let

$$(6.22) \quad \Delta : \widehat{\mathcal{B}}_n(\widehat{\mathbb{C}}) \rightarrow \mathbb{R}, \quad \alpha \mapsto \mathfrak{R}_n \circ \widehat{\mathcal{L}}_n(\alpha) - \mathcal{L}_n \circ r(\alpha).$$

It then follows from Theorem 2.10 that (where $m: \mathbb{R} \otimes \mathbb{R} \rightarrow \mathbb{R}$ is multiplication)

$$(6.23) \quad \Delta = m \circ \left(\sum_{s=1}^{n-2} (\mathfrak{R}_{n-s} \circ \widehat{\mathcal{L}}_{n-s}) \otimes C_s \right) \circ \Psi_s + (\det \circ D) \circ \Psi_{n-2}.$$

By (6.19) this vanishes on $\text{Ker}(\delta)$, and the result follows.

7. A LIFT OF GONCHAROV'S REGULATOR

We begin with a review of Goncharov's results [Gon94] (revised in [Gon96]; see also [Gon05b, Gon05a]).

7.1. Goncharov's regulator. For positive integers $2 \leq p < q$, let $\widetilde{\text{Gr}}(p, q)$ denote the affine cone over the Grassmannian of p -planes in q -space. An element can be represented by a $p \times q$ matrix defined up to the action by $\text{SL}(p)$ and we thus have an action of S_q (the symmetric group on q letters) on $\widetilde{\text{Gr}}(p, q)$ obtained by permuting the columns of a representing matrix. For any p -element subset I of $\{1, \dots, q\}$ we have a *Plücker coordinate* a_I defined as the $p \times p$ minor determined by I . Let $\widetilde{\text{Gr}}(p, q)^*$ denote the points where all Plücker coordinates are non-zero. Goncharov showed that there is a commutative diagram

$$(7.1) \quad \begin{array}{ccccccc} \mathbb{Z}[\widetilde{\text{Gr}}(3, 7)^*(\mathbb{C})] & \xrightarrow{\partial} & \mathbb{Z}[\widetilde{\text{Gr}}(3, 6)^*(\mathbb{C})] & \xrightarrow{\partial} & \mathbb{Z}[\widetilde{\text{Gr}}(3, 5)^*(\mathbb{C})] & \xrightarrow{\partial} & \mathbb{Z}[\widetilde{\text{Gr}}(3, 4)^*(\mathbb{C})] \\ \downarrow & & \downarrow g_5 & & \downarrow g_4 & & \downarrow g_3 \\ 0 & \longrightarrow & \mathcal{B}_3(\mathbb{C})_{\mathbb{Q}} & \xrightarrow{\delta} & (\mathcal{B}_2(\mathbb{C}) \otimes \mathbb{C}^*)_{\mathbb{Q}} & \xrightarrow{\delta} & \wedge^3(\mathbb{C}^*)_{\mathbb{Q}} \end{array}$$

where the boundary maps ∂ are the simplicial ones, and where

$$(7.2) \quad \begin{aligned} g_3 &= \frac{1}{6} \text{Alt}_4(a_{134} \wedge a_{124} \wedge a_{123}), \\ g_4 &= \frac{1}{12} \text{Alt}_5 \left([r(\bar{v}_1 | \bar{v}_2, \bar{v}_3, \bar{v}_4, \bar{v}_5)] \otimes a_{123} \right), \\ g_5 &= \frac{1}{90} \text{Alt}_6 \left(\left[\begin{array}{ccc} a_{124} a_{235} a_{136} \\ a_{125} a_{236} a_{134} \end{array} \right] \right). \end{aligned}$$

Here, a quadruple of points in $(x_1, x_2, x_3, x_4) \in P_{\mathbb{C}}^1 = \mathbb{C} \cup \{\infty\}$ has a *cross-ratio* $\frac{(x_1-x_3)(x_2-x_4)}{(x_1-x_4)(x_2-x_3)}$, and $r(\bar{v}_1 | \bar{v}_2, \bar{v}_3, \bar{v}_4, \bar{v}_5)$ denotes the cross-ratio of the projection of the quadruple (v_2, v_3, v_4, v_5) to $P(\mathbb{C}^3 / \langle v_1 \rangle) = P_{\mathbb{C}}^1$. Also, $\text{Alt}_n([x]) = \sum_{\sigma \in S_n} \text{sgn}(\sigma) [\sigma(x)]$.

Letting $G_q(p) = \mathbb{Z}[\widetilde{\text{Gr}}(p, q+1)^*]$, there is a canonical map $\Gamma: H_*(\text{SL}(p, \mathbb{C})) \rightarrow H_*(G_*(p))$, and Goncharov showed that the composition

$$(7.3) \quad H_5(\text{SL}(3, \mathbb{C})) \xrightarrow{\Phi} H_5(G_*(3)) \xrightarrow{g_5} H^1(\Gamma(\mathbb{C}, 3))_{\mathbb{Q}} \xrightarrow{\mathcal{L}_3} \mathbb{R}$$

is a non-zero rational multiple of the Borel regulator. Defining $\Gamma_i(\mathbb{C}, n) = \Gamma^{2n-i}(\mathbb{C}, n)$ one may view (7.1) as a chain map $G_*(3) \rightarrow \Gamma_*(\mathbb{C}, 3)_{\mathbb{Q}}$.

7.2. Cluster ensembles and differential $\widehat{\mathcal{L}}_n$ relations. The varieties $\widetilde{\text{Gr}}(p, q)$ are cluster ensembles in the sense of Fock and Goncharov [FG09]. The only thing we shall need about cluster ensembles is that there are two types of coordinates called \mathcal{A} -coordinates and \mathcal{X} -coordinates. The \mathcal{A} -coordinates are regular functions and include the Plücker coordinates. The \mathcal{X} -coordinates are monomial expressions in the \mathcal{A} -coordinates satisfying that if X is an \mathcal{X} -coordinate, then $1 + X$ has a canonical factorization as a monomial in the \mathcal{A} -coordinates. One may think of them as

generalizations of cross-ratios. We note that $\widetilde{\text{Gr}}(p, q)$ has finitely many \mathcal{A} and \mathcal{X} coordinates if and only if $(p-2)(p+q-2) < 4$ (see e.g. [?, GGS⁺13]).

If we regard an \mathcal{A} -coordinate as a formal variable, each \mathcal{X} -coordinate determines a generator $[X, 1+X]$ of $\widetilde{\mathcal{S}}_1 \times \widetilde{\mathcal{S}}_1$ (using multiplicative notation; see Section 3.5). Given a finite collection C of \mathcal{X} -coordinates of $\widetilde{\text{Gr}}(p, q)$, it becomes a simple linear algebra problem to determine if

$$(7.4) \quad \alpha = \sum_{X \in C} r_X [X, 1+X] \in \mathbb{Z}[\widetilde{\mathcal{S}}_1 \times \widetilde{\mathcal{S}}_1]$$

is a differential $\widehat{\mathcal{L}}_n$ relation. One always has the *inversion relations*

$$(7.5) \quad [X, 1+X] + (-1)^n [X^{-1}, \frac{1+X}{X}]$$

but when $n > 3$ there seems to be no other relations of the form (7.4).

Remark 7.1. Since we are primarily interested in realizations in $\mathbb{Z}[\widehat{\mathbb{C}}]$, we may consider $[\frac{X}{1+X}, \frac{1}{1+X}]$ instead of $[X, 1+X]$. We believe that this is in fact more natural.

Example 7.2. For $\widetilde{\text{Gr}}(3, 6)$ there are 22 \mathcal{A} -coordinates, the 20 Plücker coordinates as well as 2 additional coordinates

$$(7.6) \quad y_1 = \det(v_1 \times v_2, v_3 \times v_4, v_5 \times v_6), \quad y_2 = \det(v_2 \times v_3, v_4 \times v_5, v_6 \times v_1),$$

where the v_i are the columns of a representing matrix. There are 104 \mathcal{X} -coordinates, which can all be obtained from the six \mathcal{X} -coordinates

$$(7.7) \quad \frac{a_{136}a_{235}}{a_{356}a_{123}}, \quad \frac{a_{126}a_{145}}{a_{124}a_{156}}, \quad \frac{a_{156}a_{236}a_{345}}{a_{136}a_{235}a_{456}}, \quad \frac{a_{123}a_{156}}{a_{126}a_{135}}, \quad \frac{a_{136}a_{145}a_{235}}{a_{123}a_{156}a_{345}}, \quad \frac{a_{123}a_{456}}{y_1}$$

by inversion $x \mapsto x^{-1}$ and the action by the (dihedral) group generated by

$$(7.8) \quad \sigma = (1, 2, 3, 4, 5, 6) \in S_6, \quad \tau = (1, 6)(2, 5)(3, 4) \in S_6.$$

Note that τ fixes y_1 and y_2 and σ flips them. The number of elements in the $\langle \sigma, \tau \rangle$ -orbits of the six \mathcal{X} -coordinates in (7.7) are 12, 12, 12, 6, 6, and 4, respectively.

For the \mathcal{X} -coordinates in (7.7), $1+X$ is given by

$$(7.9) \quad \frac{a_{135}a_{236}}{a_{123}a_{356}}, \quad \frac{a_{125}a_{146}}{a_{124}a_{156}}, \quad \frac{a_{356}y_2}{a_{136}a_{235}a_{456}}, \quad \frac{a_{125}a_{136}}{a_{126}a_{135}}, \quad \frac{a_{135}y_2}{a_{123}a_{156}a_{345}}, \quad \frac{a_{124}a_{356}}{y_1}.$$

Up to the inversion relations (7.5) there are 25 linearly independent $\widehat{\mathcal{L}}_2$ relations and a single 40 term differential $\widehat{\mathcal{L}}_3$ relation R_{40} . The 40 term relation is a lift of the 40 term relation for \mathcal{L}_3 found in [GGS⁺13]. All lower levels are killed modulo 6-torsion.

7.3. A lift of Goncharov's regulator. Let $\widetilde{\text{Gr}}(p, q)^{\mathcal{A} \neq 0}(\mathbb{C})$ denote the points with non-zero \mathcal{A} -coordinates, and fix a branch of logarithm. Regarding the \mathcal{A} -coordinates as formal variables, and letting $X_{p,q}$ be the set of \mathcal{X} -coordinates of $\widetilde{\text{Gr}}(p, q)$, we may regard an element of $\mathbb{Z}[X_{p,q}]$ either as an element in $\mathbb{Z}[\widetilde{\mathcal{S}}_1 \times \widetilde{\mathcal{S}}_1]$ or as a map $\mathbb{Z}[\widetilde{\text{Gr}}(p, q)^{\mathcal{A} \neq 0}(\mathbb{C})] \rightarrow \mathbb{Z}[\widehat{\mathbb{C}}]$. Consider the element

$$(7.10) \quad \eta = \text{Alt}_{\langle \sigma^2, \tau \rangle} \left(\left[\frac{a_{146}a_{245}}{a_{145}a_{246}}, \frac{a_{124}a_{456}}{a_{145}a_{246}} \right] + \left[\frac{a_{124}a_{456}}{a_{145}a_{246}}, \frac{a_{146}a_{245}}{a_{145}a_{246}} \right] + \left[\frac{a_{123}a_{146}a_{245}}{a_{124}y_2}, \frac{a_{126}a_{145}a_{234}}{a_{124}y_2} \right] \right. \\ \left. + \left[\frac{a_{126}a_{145}a_{234}}{a_{124}y_2}, \frac{a_{123}a_{146}a_{245}}{a_{124}y_2} \right] + \left[\frac{a_{124}a_{156}}{a_{125}a_{146}}, \frac{a_{126}a_{145}}{a_{125}a_{146}} \right] + \left[\frac{a_{126}a_{145}}{a_{125}a_{146}}, \frac{a_{124}a_{156}}{a_{125}a_{146}} \right] \right) \\ - \text{Alt}_{\langle \tau \rangle} \left(\left[\frac{a_{126}a_{234}a_{456}}{a_{246}y_2}, \frac{a_{146}a_{236}a_{245}}{a_{246}y_2} \right] + \left[\frac{a_{146}a_{236}a_{245}}{a_{246}y_2}, \frac{a_{126}a_{234}a_{456}}{a_{246}y_2} \right] \right).$$

Theorem 7.3. *There is a commutative diagram*

$$(7.11) \quad \begin{array}{ccccccc} \mathbb{Z}[\widetilde{\text{Gr}}(3, 7)^{A \neq 0}(\mathbb{C})] & \xrightarrow{\partial} & \mathbb{Z}[\widetilde{\text{Gr}}(3, 6)^{A \neq 0}(\mathbb{C})] & \xrightarrow{\partial} & \mathbb{Z}[\widetilde{\text{Gr}}(3, 5)^*(\mathbb{C})] & \xrightarrow{\partial} & \mathbb{Z}[\widetilde{\text{Gr}}(3, 4)^*(\mathbb{C})] \\ \downarrow & & \downarrow f_5 & & \downarrow f_4 & & \downarrow f_3 \\ 0 & \longrightarrow & \widehat{\mathcal{B}}_3(\widehat{\mathbb{C}}) & \xrightarrow{\delta} & (\widehat{\mathcal{B}}_2(\widehat{\mathbb{C}}) \otimes \mathbb{C}) & \xrightarrow{\delta} & \wedge^3(\mathbb{C}) \end{array}$$

with maps defined by

$$(7.12) \quad \begin{aligned} f_3 &= \text{Alt}_{\langle(1,2,3,4)\rangle}(\widetilde{a}_{134} \wedge \widetilde{a}_{124} \wedge \widetilde{a}_{123}), \\ f_4 &= -\text{Alt}_{\langle(1,2,3,4,5)\rangle} \left(\left[\frac{a_{125}a_{134}}{a_{124}a_{135}}, \frac{a_{123}a_{145}}{a_{124}a_{135}} \right] \otimes (\widetilde{a}_{123} + \widetilde{a}_{145}) \right), \\ f_5 &= \eta. \end{aligned}$$

If $G_*^{A \neq 0}(3)$ denotes the top chain complex, the composition

$$(7.13) \quad H_5(G_*^{A \neq 0}(3)) \xrightarrow{f_5} H^1(\widehat{\Gamma}(\mathbb{C}, 3)) \xrightarrow{r} \mathcal{B}_3(\mathbb{C})_{\mathbb{Q}}$$

agrees with Goncharov's map g_5 .

Proof. The proof that $\delta f_4 = f_3 \partial$ is elementary. We next show that $\delta f_5 = f_4 \partial$. We have

$$(7.14) \quad \delta f_5 - f_4 \partial = \sum A_a \otimes \widetilde{a},$$

where the sum is over the Plücker-coordinates and y_2 . For example, we have

$$(7.15) \quad \begin{aligned} A_{a_{124}} &= \left[\frac{a_{124}a_{456}}{a_{145}a_{246}}, \frac{a_{146}a_{245}}{a_{145}a_{246}} \right] - \left[\frac{a_{123}a_{146}a_{245}}{a_{124}y_2}, \frac{a_{126}a_{145}a_{234}}{a_{124}y_2} \right] - \left[\frac{a_{126}a_{145}a_{234}}{a_{124}y_2}, \frac{a_{123}a_{146}a_{245}}{a_{124}y_2} \right] \\ &+ \left[\frac{a_{124}a_{156}}{a_{125}a_{146}}, \frac{a_{126}a_{145}}{a_{125}a_{146}} \right] + \left[\frac{a_{123}a_{245}}{a_{124}a_{235}}, \frac{a_{125}a_{234}}{a_{124}a_{235}} \right] + \left[\frac{a_{125}a_{234}}{a_{124}a_{235}}, \frac{a_{123}a_{245}}{a_{124}a_{235}} \right] \\ &+ \left[\frac{a_{146}a_{245}}{a_{145}a_{246}}, \frac{a_{124}a_{456}}{a_{145}a_{246}} \right] + \left[\frac{a_{126}a_{145}}{a_{125}a_{146}}, \frac{a_{124}a_{156}}{a_{125}a_{146}} \right], \end{aligned}$$

which is a sum of instances of $[A, B] + [B, A]$. One can show (see Zickert [Zic15, Rm. 8.7]) that $24([(u, v)] + [(u, v)])$ is a consequence of the lifted 5-term relations, so it follows that $A_{a_{124}} \otimes \widetilde{a}_{124} = 0 \in \widehat{\mathcal{B}}_2(\widehat{\mathbb{C}}) \otimes \mathbb{C}$. Similarly, $A_a \otimes \widetilde{a}$ is zero for the \mathcal{A} -coordinates $a_{125}, a_{134}, a_{256}, a_{346}, a_{356}, a_{135}, a_{246}$, and y_2 . For the remaining terms we obtain

$$(7.16) \quad \delta f_5 - f_4 \partial = \text{Alt}_{\langle\sigma^2, \tau\rangle} (A_{a_{123}} \otimes \widetilde{a}_{123} + A_{a_{136}} \otimes \widetilde{a}_{136}).$$

One then checks that $A_{a_{123}}$ and $A_{a_{136}}$ are (up to instances of $[A, B] + [B, A]$) a sum of lifted five term relations. For example, we have $r(A_{a_{123}}) = R_1 + R_2 + R_3$, where

$$(7.17) \quad \begin{aligned} R_1 &= \left[\frac{a_{123}a_{345}}{a_{134}a_{235}} \right] + \left[\frac{a_{125}a_{345}}{a_{135}a_{245}} \right] + \left[\frac{a_{125}a_{234}}{a_{124}a_{235}} \right] - \left[\frac{a_{124}a_{345}}{a_{134}a_{245}} \right] - \left[\frac{a_{125}a_{134}}{a_{124}a_{135}} \right] \\ R_2 &= -\left[\frac{a_{126}a_{345}}{a_{136}a_{245}} \right] - \left[\frac{a_{126}a_{145}a_{234}}{a_{124}y_2} \right] + \left[\frac{a_{124}a_{345}}{a_{134}a_{245}} \right] - \left[\frac{a_{123}a_{146}a_{345}}{a_{134}y_2} \right] + \left[\frac{a_{126}a_{134}}{a_{124}a_{136}} \right] \\ R_3 &= -\left[\frac{a_{126}a_{135}}{a_{125}a_{136}} \right] + \left[\frac{a_{123}a_{156}a_{345}}{a_{135}y_2} \right] + \left[\frac{a_{126}a_{345}}{a_{136}a_{245}} \right] - \left[\frac{a_{125}a_{345}}{a_{135}a_{245}} \right] - \left[\frac{a_{123}a_{156}a_{245}}{a_{125}y_2} \right]. \end{aligned}$$

It is not difficult to check that these are (up to instances of $[x] + [1-x]$) five term relations. This concludes the proof that $\delta f_5 = f_4 \partial$.

We now prove that $f_5\partial = 0$. To see this we first compute $w_3(\eta)$ (recall Definition 3.7). A straightforward computation shows that $w_3(\eta) = \text{Alt}_{\langle\sigma\rangle}(\phi)$, where

$$(7.18) \quad w_3(\eta) = \text{Alt}_{\langle\sigma\rangle}(\phi), \quad \phi = -\text{Alt}_{\langle(1,2,3,4,5)\rangle} \left(w_2 \left(\left[\frac{a_{125}a_{134}}{a_{124}a_{135}}, \frac{a_{123}a_{145}}{a_{124}a_{135}} \right] \right) (\tilde{a}_{123} + \tilde{a}_{145}) \right)$$

From this we conclude that $w_3(f_5\partial) = 0$. Since $\delta f_5 = f_4\partial$ it follows that $f_5\partial$ kills lower levels modulo instances of $[A, B] + [B, A]$. In fact, these terms cancel out, so $f_5\partial$ kills lower levels. One easily checks that $f_5\partial$ has proper ambiguity, so $f_5\partial$ is constant in $\widehat{\mathcal{B}}_3(\widehat{\mathbb{C}})$. One then checks that $f_5\partial$ vanishes on the nose for the element

$$(7.19) \quad \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & -2 & 1 \\ 0 & 1 & 0 & 1 & -2 & 1 & -\frac{7}{8} \\ 0 & 0 & 1 & \frac{1}{4} & 1 & 1 & -\frac{7}{2} \end{pmatrix}.$$

This concludes the proof that $f_5\partial = 0$. To see that $r(f_5)$ agrees with g_5 in homology, it is enough to prove that $\frac{1}{720} \text{Alt}_6(r(\eta)) = f_5 \in \mathcal{B}_3(\mathbb{C})_{\mathbb{Q}}$. Since $[x] + [1-x] = -[-\frac{1-x}{x}] + [1] \in \mathcal{B}_3(\mathbb{C})$ we have

$$(7.20) \quad r(\eta) = -\text{Alt}_{\langle\sigma^2, \tau\rangle} \left(\left[-\frac{a_{124}a_{456}}{a_{146}a_{245}} \right] + \left[-\frac{a_{126}a_{145}a_{234}}{a_{123}a_{146}a_{245}} \right] + \left[-\frac{a_{126}a_{145}}{a_{124}a_{156}} \right] \right) + \text{Alt}_{\langle\tau\rangle} \left(\left[-\frac{a_{146}a_{236}a_{245}}{a_{126}a_{234}a_{456}} \right] \right).$$

The fact that $\frac{1}{720} \text{Alt}_6(r(\eta)) = f_5 \in \mathcal{B}_3(\mathbb{C})_{\mathbb{Q}}$ can now be verified by a term by term comparison, which does not use any relations in $\mathcal{B}_3(\mathbb{C})_{\mathbb{Q}}$. \square

Remark 7.4. It follows that $r(f_5)$ is equivalent to Goncharov's formula, but has the advantage of being defined with integral coefficients and without symmetrization.

Remark 7.5. It follows from (7.18) that $\eta + \sigma(\eta)$ is a differential $\widehat{\mathcal{L}}_3$ relation with 80 terms. It has proper ambiguity and all lower levels are killed on the nose. It vanishes for the matrix (7.19) with the last column removed, so is identically 0. From this it follows that f_5 is skew symmetric under the action by the dihedral group.

Remark 7.6. The diagram (7.11) may be defined over an arbitrary field. The righthand square always commutes, the middle square commutes modulo 24 torsion, and the left square commutes for any field where all \mathcal{A} -coordinates of (7.19) are defined and non-zero.

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