THE PTOLEMY FIELD OF 3-MANIFOLD-REPRESENTATIONS

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Abstract. The Ptolemy coordinates for boundary-unipotent $\mathrm{SL}(n,\mathbb{C})$-representations of a 3-manifold group were introduced in [7] inspired by the $\mathcal{A}$-coordinates on higher Teichmüller space due to Fock and Goncharov. In this paper, we define the Ptolemy field of a (generic) $\mathrm{PSL}(2,\mathbb{C})$-representation and prove that it coincides with the trace field of the representation. This gives an efficient algorithm to compute the trace field of a cusped hyperbolic manifold.

Contents

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1. Introduction

1.1. **The Ptolemy coordinates.** The Ptolemy coordinates for boundary-unipotent representations of a 3-manifold group in $\mathrm{SL}(n,\mathbb{C})$ were introduced in Garoufalidis–Thurston–Zickert [7] inspired by the $\mathcal{A}$-coordinates on higher Teichmüller space due to Fock and Goncharov [4]. In this paper we will focus primarily on representations in $\mathrm{SL}(2,\mathbb{C})$ and $\mathrm{PSL}(2,\mathbb{C})$.

Given a topological ideal triangulation $\mathcal{T}$ of an oriented compact 3-manifold M, a *Ptolemy assignment* (for $\mathrm{SL}(2,\mathbb{C})$) is an assignment of a non-zero complex number (called a *Ptolemy coordinate*) to each 1-cell of $\mathcal{T}$, such that for each simplex, the Ptolemy coordinates assigned to the edges ε_{ij} satisfy the *Ptolemy relation*

$$c_{03}c_{12} + c_{01}c_{23} = c_{02}c_{13}. \tag{1.1}$$

The set of Ptolemy assignments is thus an affine variety $P_2(\mathcal{T})$, which is cut out by homogeneous quadratic polynomials.

In this paper we define the Ptolemy field of a boundary-unipotent representation and show that it is isomorphic to the trace field. This gives rise to an efficient algorithm for *exact* computation of the trace field of a hyperbolic manifold.

1.2. **Decorated** $\mathrm{SL}(2,\mathbb{C})$**-representations.** The precise relationship between Ptolemy assignments and representations is given by

$$\left\{\begin{matrix}\text{Points in}\\ P_2(\mathcal{T})\end{matrix}\right\} \xleftrightarrow{1-1} \left\{\begin{matrix}\text{Natural } (\mathrm{SL}(2,\mathbb{C}),P)\text{-}\\ \text{cocycles on } M\end{matrix}\right\} \xleftrightarrow{1-1} \left\{\begin{matrix}\text{Generically decorated}\\ (\mathrm{SL}(2,\mathbb{C}),P)\text{-representations}\end{matrix}\right\}. \tag{1.2}$$

The concepts are briefly described below, and the correspondences are illustrated in Figures 5 and 3. We refer to Section 2 for a summary of our notation. The bijections of (1.2) first appeared in Zickert [12] (in a slightly different form), and were generalized to $\mathrm{SL}(n,\mathbb{C})$-representations in Garoufalidis–Thurston–Zickert [7].

- *Natural cocycle:* Labeling of the edges of each truncated simplex by elements in $\mathrm{SL}(2,\mathbb{C})$ satisfying the cocycle condition (the product around each face is 1). The long edges are counter diagonal, i.e. of the form $\left(\begin{smallmatrix}0 & -x^{-1}\\ x & 0\end{smallmatrix}\right)$, and the short edges are *non-trivial* elements in P. Identified edges are labeled by the same group element.
- *Decorated representation:* A *decoration* of a boundary-parabolic representation ρ is an assignment of a coset gP to each vertex of $\widehat{\widetilde{M}}$ which is equivariant with respect to ρ. A decoration is *generic* if for each edge joining two vertices, the two P-cosets gP, hP are distinct as B-cosets. This condition is equivalent to $\det(ge_1, he_1) \neq 0$.

Two decorations are considered equal if they differ by left multiplication by a group element g.

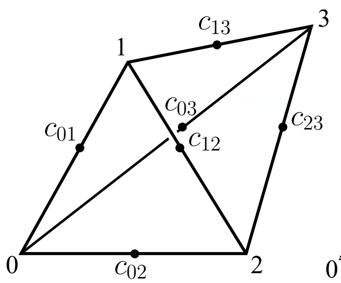

Figure 1. Ptolemy assignment; the Ptolemy relation (1.1) holds.

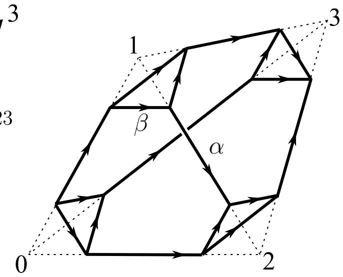

Figure 2. Natural cocycle; α is counter-diagonal, $\beta \in P$.

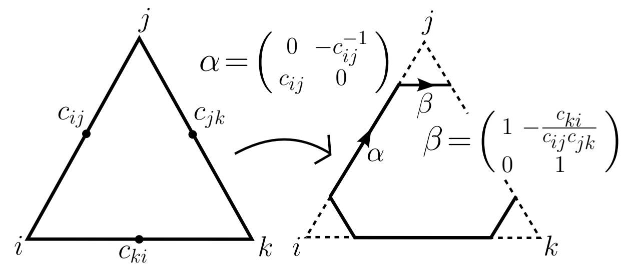

Figure 3. From Ptolemy assignments to natural cocycles.

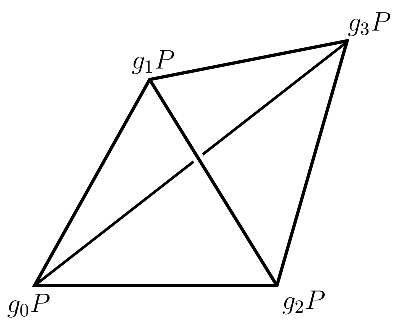

Figure 4. Decoration; equivariant assignment of cosets.

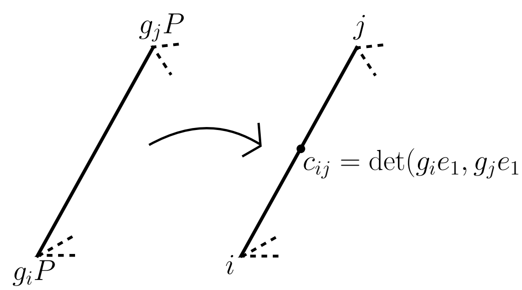

Figure 5. From decorations to Ptolemy assignments.

By ignoring the decoration, (1.2) yields a map

$$\mathcal{R}\colon P_2(\mathcal{T}) \to \big\{(\mathrm{SL}(2,\mathbb{C}), P)\text{-representations}\big\}\big/\operatorname{Conj}. \tag{1.3}$$

The representation corresponding to a Ptolemy assignment is given explicitly in terms of the natural cocycle.

Remark 1.1. Note that a natural cocycle canonically determines a representation of the edge path groupoid of the triangulation of M by truncated simplices.

Remark 1.2. Every boundary-parabolic representation has a decoration, but a representation may have only non-generic decorations. The map $\mathcal{R}$ is thus not surjective in general, and the image depends on the triangulation. However, if the triangulation is sufficiently fine, $\mathcal{R}$ is surjective (see [7]). The preimage of a representation depends on the image of the peripheral subgroups (see Proposition 1.9).

1.3. **Obstruction classes and** $\mathrm{PSL}(2,\mathbb{C})$**-representations.** There is a subtle distinction between representations in $\mathrm{SL}(2,\mathbb{C})$ versus $\mathrm{PSL}(2,\mathbb{C})$. The geometric representation of a hyperbolic manifold always lifts to an $\mathrm{SL}(2,\mathbb{C})$-representation, but for a one-cusped manifold, no lift is boundary-parabolic (any lift will take a longitude to an element of trace -2 [2]).

The obstruction to lifting a boundary-parabolic $\mathrm{PSL}(2,\mathbb{C})$-representation to a boundary-parabolic $\mathrm{SL}(2,\mathbb{C})$-representation is a class in $H^2(\widehat{M};\mathbb{Z}/2\mathbb{Z})$. For each such class there is a Ptolemy variety $P_2^\sigma(\mathcal{T})$, which maps to the set of $\mathrm{PSL}(2,\mathbb{C})$-representations with obstruction class σ. More precisely, $P_2^\sigma(\mathcal{T})$ is defined for each 2-cocycle $\sigma\in Z^2(\widehat{M};\mathbb{Z}/2\mathbb{Z})$, and up to canonical isomorphism only depends on the cohomology class of σ. The Ptolemy variety for the trivial cocycle equals $P_2(\mathcal{T})$. The analogue of (1.2) is

$$\left\{\begin{matrix}\text{Points in}\\ P_2^\sigma(\mathcal{T})\end{matrix}\right\} \overset{1-1}{\longleftrightarrow} \left\{\begin{matrix}\text{Lifted natural}\\ (\mathrm{PSL}(2,\mathbb{C}),P)\text{-cocycles}\\ \text{with obstruction cocycle } \sigma\end{matrix}\right\} \twoheadrightarrow \left\{\begin{matrix}\text{Generically decorated}\\ (\mathrm{PSL}(2,\mathbb{C}),P)\text{-representations}\\ \text{with obstruction class } \sigma\end{matrix}\right\}. \tag{1.4}$$

A lifted natural cocycle is defined as above, except that the product along a face is now $\pm I$, where the sign is determined by σ. The right map is no longer a 1-1 correspondence; the preimage of each decorated representation is the choice of lifts, i.e. parametrized by a cocycle in $Z^1(\widehat{M};\mathbb{Z}/2\mathbb{Z})$. We refer to [7] for details. As in (1.3), ignoring the decoration yields a map

$$\mathcal{R}\colon P_2^\sigma(\mathcal{T}) \to \left\{\begin{matrix}(\mathrm{PSL}(2,\mathbb{C}),P)\text{-representations}\\ \text{with obstruction class } \sigma\end{matrix}\right\} \Big/ \operatorname{Conj}. \tag{1.5}$$

which is explictly given in terms of the natural cocycle.

Theorem 1.3 (Garoufalidis–Thurston–Zickert [7])**.** *If M is hyperbolic, and all edges of $\mathcal{T}$ are essential, the geometric representation is in the image of $\mathcal{R}$.*

Remark 1.4. If $\mathcal{T}$ has a non-essential edge, all Ptolemy varieties will be empty. Hence, if $P_2^\sigma(\mathcal{T})$ is non-empty for some σ, and if M is hyperbolic, the geometric representation is detected by the Ptolemy variety of the geometric obstruction class.

1.4. **Our results.** We view the Ptolemy varieties $P_2^\sigma(\mathcal{T})$ as subsets of an ambient space $\mathbb{C}^e$, with coordinates indexed by the 1-cells of $\mathcal{T}$. Let $T=(\mathbb{C}^*)^v$, with the coordinates indexed by the boundary components of M.

Definition 1.5. The *diagonal action* is the action of T on $P_2^\sigma(\mathcal{T})$ where $(x_1,\dots,x_v)\in T$ acts on a Ptolemy assignment by replacing the Ptolemy coordinate c of an edge e with x_ix_jc, where x_i and x_j are the coordinates corresponding to the ends of e. Let

$$P_2^\sigma(\mathcal{T})_{\mathrm{red}} = P_2^\sigma(\mathcal{T})/T. \tag{1.6}$$

Definition 1.6. A boundary-parabolic $\mathrm{PSL}(2,\mathbb{C})$-representation is *generic* if it has a generic decoration. It is *boundary-non-trivial* if each peripheral subgroup has non-trivial image.

Remark 1.7. Note that the notion of genericity is with respect to the triangulation. By Theorem 1.3, if all edges of $\mathcal{T}$ are essential (and $\mathcal{T}$ has no interior vertices), the geometric representation of a cusped hyperbolic manifold is always generic and boundary-non-trivial.

Remark 1.8. Note that if M has spherical boundary components (e.g. if $\mathcal{T}$ is a triangulation of a closed manifold), no representation is boundary-non-trivial.

Proposition 1.9. The map $\mathcal{R}$ in (1.5) factors through $P_2^\sigma(\mathcal{T})_{\mathrm{red}}$, i.e., we have

$$\mathcal{R}\colon P_2^\sigma(\mathcal{T})_{\mathrm{red}} \longrightarrow \left\{\begin{matrix}(\mathrm{PSL}(2,\mathbb{C}),P)\text{-representations}\\ \text{with obstruction class } \sigma\end{matrix}\right\} \Big/ \mathrm{Conj}\,. \tag{1.7}$$

The image is the set of generic representations, and the preimage of a generic, boundary-non-trivial representation is finite and parametrized by $H^1(\widehat{M};\mathbb{Z}/2\mathbb{Z})$.

Remark 1.10. For the corresponding map from $P_2(\mathcal{T})_{\mathrm{red}}$ to $(\mathrm{SL}(2,\mathbb{C}),P)$-representations, the preimage of a generic boundary-non-trivial representation is a single point.

Remark 1.11. The preimage of a representation which is not boundary-non-trivial is never finite. In fact its dimension is the number of boundary-components that are collapsed. In particular, it follows that if $c \in P_2^\sigma(\mathcal{T})_{\mathrm{red}}$ is in a 0-dimensional component (which is not contained in a higher dimensional component), the image is boundary-non-trivial.

By geometric invariant theory, $P_2^\sigma(\mathcal{T})_{\mathrm{red}}$ is a variety whose coordinate ring is the ring of invariants $\mathcal{O}^T$ of the coordinate ring $\mathcal{O}$ of $P_2^\sigma(T)$.

Definition 1.12. Let $c \in P_2^\sigma(\mathcal{T})$. The Ptolemy field of c is the field

$$k_c = \mathbb{Q}\left(\left\{p(c_1,\dots,c_e) \mid p \in \mathcal{O}^T\right\}\right). \tag{1.8}$$

The Ptolemy field of a generic boundary-non-trivial representation is the Ptolemy field of any preimage under (1.7).

Clearly, the Ptolemy field only depends on the image in $P_2^\sigma(\mathcal{T})_{\mathrm{red}}$. Our main result is the following.

Theorem 1.13. *The Ptolemy field of a boundary-non-trivial, generic, boundary-parabolic representation ρ in* $\mathrm{PSL}(2,\mathbb{C})$ *or* $\mathrm{SL}(2,\mathbb{C})$ *is equal to its trace field.* □

Remark 1.14. For a cusped hyperbolic 3-manifold the *shape field* is in general smaller than the trace field. The shape field equals the *invariant trace field* (see, e.g., [10]).

For computations of the Ptolemy field, we need an explicit description of the ring of invariants $\mathcal{O}^T$, or equivalently, the reduced Ptolemy variety $P_2^\sigma(\mathcal{T})_{\mathrm{red}}$.

Proposition 1.15. There exist 1-cells $\varepsilon_1,\dots,\varepsilon_v$ of $\mathcal{T}$ such that the reduced Ptolemy variety $P_2^\sigma(\mathcal{T})_{\mathrm{red}}$ is naturally isomorphic to the subvariety of $P_2^\sigma(\mathcal{T})$ obtained by intersecting with the affine hyperplane $c_{\varepsilon_1} = \dots = c_{\varepsilon_v} = 1$.

Corollary 1.16. Let $c \in P_2^\sigma(\mathcal{T})_{\mathrm{red}}$. Under an isomorphism as in Proposition 1.15, the Ptolemy field of c is the field generated by the Ptolemy coordinates. □

Remark 1.17. A concrete method for selecting 1-cells as in Proposition 1.15 is described in Section 4.3.

Analogues of our results for higher rank Ptolemy varieties are discussed in Section 6. The analogue of Proposition 1.9 holds for representations that are *boundary-non-degenerate* (see Definition 6.10), and the analogue of Proposition 1.15 leads to a simple computation of the Ptolemy field.

Conjecture 1.18. The Ptolemy field of a boundary-non-degenerate, generic, boundary-unipotent representation ρ in $\mathrm{SL}(n,\mathbb{C})$ or $\mathrm{PSL}(n,\mathbb{C})$ is equal to its trace field. □

Remark 1.19. The computation of reduced Ptolemy varieties is remarkably efficient using Magma [1]. For all but a few census manifolds, primary decompositions of the (reduced) Ptolemy varieties $P_2^\sigma(\mathcal{T})$ can be computed in a fraction of a second on a standard laptop. A database of representations can be found at `http://ptolemy.unhyperbolic.org/` [9]. All our tools have been incorporated into SnapPy [3] by the second author and the Ptolemy fields obtained through:

```
>>> from snappy import Manifold
>>> p=Manifold("m019").ptolemy_variety(2,'all')
>>> p.retrieve_solutions().number_field()
... [[x^4 - 2*x^2 - 3*x - 1], [x^4 + x - 1]]
```

The number fields are grouped by obstruction class. In this example, we see that the Ptolemy variety for the non-trivial obstruction class has a component with number field $x^4 + x - 1$, which is the trace field of m019. The above code retrieves a precomputed decomposition of the Ptolemy variety from [9]. In sage or SnapPy with Magma installed, you can use `p.compute_solutions().number_field()` to compute the decomposition.

2. Notation

2.1. **Triangulations.** Let M be a compact oriented 3-manifold with (possibly empty) boundary. We refer to the boundary components as *cusps* (although they may not be tori). Let $\widehat{M}$ be the space obtained from M by collapsing each boundary component to a point.

Definition 2.1. A (concrete) *triangulation* of M is an identification of $\widehat{M}$ with a space obtained from a collection of simplices by gluing together pairs of faces by affine homeomorphisms. For each simplex Δ of $\mathcal{T}$ we fix an identification of Δ with a standard simplex.

Remark 2.2. By drilling out disjoint balls if necessary (this does not change the fundamental group), we may assume that the triangulation of M is *ideal*, i.e. that each 0-cell corresponds to a boundary-component of M. For example, we regard a triangulation of a closed manifold as an ideal triangulation of a manifold with boundary a union of spheres.

Definition 2.3. A triangulation is *oriented* if the identifications with standard simplices are orientation preserving.

Remark 2.4. All of the triangulations in the SnapPy censuses `OrientableCuspedCensus`, `LinkExteriors` and `HTLinkExteriors` [3] are oriented. Unless otherwise specified we shall assume that our triangulations are oriented.

A triangulation gives rise to a triangulation of M by truncated simplices, and to a triangulation of $\widehat{\widehat{M}}$.

2.2. Miscellaneous.

- The number of vertices, edges, faces, and simplices, of a triangulation $\mathcal{T}$ are denoted by v, e, f, and s, respectively.
- The standard basic vectors in $\mathbb{Z}^k$ are denoted by $e_1, \dots, e_k$.
- The (oriented) edge of simplex k from vertex i to j is denoted by $\varepsilon_{ij,k}$.
- The matrix groups $\{\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)\}$ and $\{\left(\begin{smallmatrix} a & x \\ 0 & a^{-1} \end{smallmatrix}\right)\}$ are denoted by P and B, respectively. The higher rank analogue of P is denoted by N.
- A representation is *boundary-parabolic* if it takes each peripheral subroup to a conjugate of P. Such is also called a (G,P)*-representation* ($G = \mathrm{SL}(2,\mathbb{C})$ or $\mathrm{PSL}(2,\mathbb{C})$). In the higher rank case, such a representation is called boundary-unipotent.
- A triangulation is *ordered* if $\varepsilon_{ij,k} \sim \varepsilon_{i'j',k'}$ implies that $i < j \iff i' < j'$.

3. The Ptolemy varieties

We define the Ptolemy variety for $n = 2$ following Garoufalidis–Thurston–Zickert [7] (see also Garoufalidis–Goerner–Zickert [5]).

3.1. **The** $\mathrm{SL}(2,\mathbb{C})$**-Ptolemy variety.** Assign to each oriented edge $\varepsilon_{ij,k}$ of $\Delta_k \in \mathcal{T}$ a *Ptolemy coordinate* $c_{ij,k}$. Consider the affine algebraic set A defined by the *Ptolemy relations*

$$c_{03,k}c_{12,k} + c_{01,k}c_{23,k} = c_{02,k}c_{13,k}, \qquad k = 1, 2, \dots, t, \tag{3.1}$$

the *identification relations*

$$c_{ij,k} = c_{i'j',k'} \qquad \text{when} \qquad \varepsilon_{ij,k} \sim \varepsilon_{i'j',k'}, \tag{3.2}$$

and the *edge orientation relations* $c_{ij,k} = -c_{ji,k}$. By only considering $i < j$, we shall always eliminate the edge orientation relations.

Definition 3.1. The *Ptolemy variety* $P_2(\mathcal{T})$ is the Zariski open subset of A consisting of points with non-zero Ptolemy coordinates.

Remark 3.2. One can concretely obtain $P_2(\mathcal{T})$ from A by adding a dummy variable x and a dummy relation $x \prod c_{ij,k} = 1$.

Remark 3.3. We can eliminate the identification relations (3.2) by selecting a representative for each edge cycle. This gives an embedding of the Ptolemy variety in an ambient space $\mathbb{C}^e$, where it is cut out by s Ptolemy relations, one for each simplex. Note that when all boundary components are tori, $s = e$.

3.1.1. *The figure-*8 *knot.* Consider the ideal triangulation of the figure-8 knot complement shown in Figure 6. The Ptolemy variety $P_2(\mathcal{T})$ is given by

$$\begin{aligned} &c_{03,0}c_{12,0} + c_{01,0}c_{23,0} = c_{02,0}c_{13,0}, \qquad && c_{02,0} = c_{12,0} = c_{13,0} = c_{01,1} = c_{03,1} = c_{23,1} \\ &c_{03,1}c_{12,1} + c_{01,1}c_{23,1} = c_{02,1}c_{13,1}, && c_{01,0} = c_{03,0} = c_{23,0} = c_{02,1} = c_{12,1} = c_{13,1}. \end{aligned} \tag{3.3}$$

By selecting representatives $\varepsilon_{23,0}$ and $\varepsilon_{13,0}$ for the 2 edge cycles, $P_2(\mathcal{T})$ embeds in $\mathbb{C}^2$ where it is given by

$$c_{23,0}c_{13,0} + c_{23,0}^2 = c_{13,0}^2, \qquad c_{13,0}c_{23,0} + c_{13,0}^2 = c_{23,0}^2. \tag{3.4}$$

It follows that $P_2(\mathcal{T})$ is empty, which is no surprise, since the only boundary-parabolic $\mathrm{SL}(2,\mathbb{C})$-representations of the figure-8 knot are abelian. To detect the geometric representation, we need to consider *obstruction classes* (see Section 3.2 below).

3.1.2. *The figure-*8 *knot sister.* Consider the ideal triangulation of the figure-8 knot sister shown in Figure 7. The Ptolemy variety $P_2(\mathcal{T})$ is given by

$$\begin{aligned} &c_{03,0}c_{12,0} + c_{01,0}c_{23,0} = c_{02,0}c_{13,0}, \qquad && c_{01,0} = -c_{03,0} = c_{23,0} = -c_{01,1} = c_{03,1} = -c_{23,1} \\ &c_{03,1}c_{12,1} + c_{01,1}c_{23,1} = c_{02,1}c_{13,1}, && c_{02,0} = -c_{12,0} = c_{13,0} = -c_{02,1} = c_{12,1} = -c_{13,1} \end{aligned} \tag{3.5}$$

Selecting representatives $\varepsilon_{23,0}$ and $\varepsilon_{13,0}$ for the 2 edge cycles, $P_2(\mathcal{T}) \in \mathbb{C}^2$ is given by

$$c_{23,0}c_{13,0} + c_{23,0}^2 = c_{13,0}^2, \qquad c_{23,0}c_{13,0} + c_{23,0}^2 = c_{13,0}^2. \tag{3.6}$$

This is equivalent to

$$x^2 - x - 1 = 0, \qquad x = \frac{c_{13,0}}{c_{23,0}}. \tag{3.7}$$

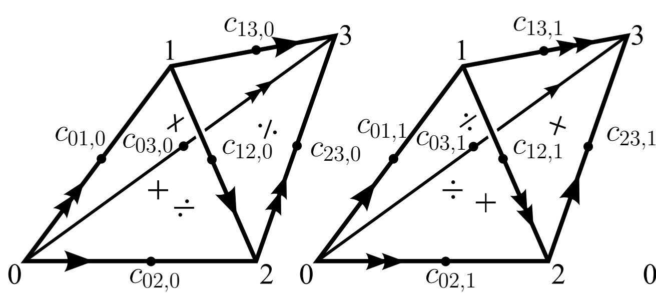

Figure 6. Ordered triangulation of the figure 8 knot. The signs indicate the non-trivial obstruction class.

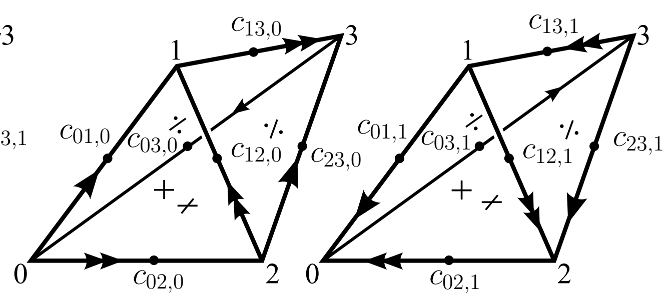

Figure 7. Oriented triangulation of the figure 8 knot sister. The signs indicate the non-trivial obstruction class.

Remark 3.4. Note that for ordered triangulations, the identification relations (3.2) do not involve minus signs. The triangulation in Figure 6 is not oriented.

3.2. **Obstruction classes.** Each class in $H^2(\widehat{M};\mathbb{Z}/2\mathbb{Z})$ can be represented by a $\mathbb{Z}/2\mathbb{Z}$-valued 2-cocycle on $\widehat{M}$, i.e. an assignment of a sign to each face of $\mathcal{T}$.

Definition 3.5. Let σ be a $\mathbb{Z}/2\mathbb{Z}$-valued 2-cocycle on $\widehat{M}$. The *Ptolemy variety* for σ is defined as in Definition 3.1, but with the Ptolemy relation replaced by

$$\sigma_{0,k}\sigma_{3,k}c_{03,k}c_{12,k} + \sigma_{0,k}\sigma_{1,k}c_{01,k}c_{23,k} = \sigma_{0,k}\sigma_{2,k}c_{02,k}c_{13,k}, \tag{3.8}$$

where $\sigma_{i,k}$ is the sign of the face of Δ_k opposite vertex i.

Remark 3.6. Multiplying σ by a coboundary $\delta(\tau)$ corresponds to multiplying the Ptolemy coordinate of a one-cell e by $\tau(e)$ (see [7] for details). Hence, up to canonical isomorphism, the Ptolemy variety $P_2^\sigma(\mathcal{T})$ only depends on the cohomology class of σ. The Ptolemy variety $P_2(\mathcal{T})$ is the Ptolemy variety for the trivial obstruction class.

3.2.1. *Examples.* In both examples above, $H^2(\widehat{M};\mathbb{Z}/2\mathbb{Z})=\mathbb{Z}/2\mathbb{Z}$, and the non-trivial obstruction class σ is indicated in Figures 6 and 7.

For the Figure-8 knot, $P_2^\sigma(\mathcal{T})$ is given by

$$-c_{23,0}c_{13,0}+c_{23,0}^2=-c_{13,0}^2,\qquad -c_{13,0}c_{23,0}+c_{13,0}^2=-c_{23,0}^2, \tag{3.9}$$

which is equivalent to

$$x^2-x+1=0,\qquad x=\frac{c_{13,0}}{c_{23,0}}. \tag{3.10}$$

The corresponding representations are the geometric representation and its conjugate.

For the Figure 8 knot sister, the Ptolemy variety becomes

$$-c_{23,0}c_{13,0}-c_{23,0}^2=c_{13,0}^2,\qquad -c_{23,0}c_{13,0}-c_{23,0}^2=c_{13,0}^2, \tag{3.11}$$

which is equivalent to

$$x^2+x+1=0,\qquad x=\frac{c_{13,0}}{c_{23,0}}. \tag{3.12}$$

4. The diagonal action

Fix an ordering of the 1-cells of $\mathcal{T}$ and of the cusps of M. As mentioned in Remark 3.3 the Ptolemy variety can be regarded as a subset of the ambient space $\mathbb{C}^e$.

Let $T=(\mathbb{C}^*)^v$ be a torus whose coordinates are indexed by the cusps of M. There is a natural action of T on $P_2^\sigma(\mathcal{T})$ defined as follows: For $x=(x_1,\dots,x_v)\in T$ and $c=(c_1,\dots c_e)\in P_2^\sigma(\mathcal{T})$ define a Ptolemy assignment cx by

$$(xc)_i=x_jx_kc_i, \tag{4.1}$$

where j and k (possibly $j=k$) are the cusps joined by the ith edge cycle. The action is thus determined entirely by the 1-skeleton of $\widehat{M}$.

Remark 4.1. There is a more intrinsic definition of this action in terms of decorations: Each vertex of $\widehat{\widetilde{M}}$ determines a cusp of M, and if D is a decoration taking a vertex w to gP, the decoration xD takes w to $g\left(\begin{smallmatrix} x_i & 0\\ 0 & x_i^{-1}\end{smallmatrix}\right)P$ where i is the cusp determined by w. The fact that the two definitions agree under the one-one correspondence (1.4) is an immediate consequence of the relationship given in Figure 5.

4.1. **The reduced Ptolemy varieties.**

Definition 4.2. The *reduced Ptolemy variety* $P_2^\sigma(\mathcal{T})_{\mathrm{red}}$ is the quotient $P_2^\sigma(\mathcal{T})/T$.

Let $\mathcal{O}$ be the coordinate ring of $P_2^\sigma(\mathcal{T})$, and let $\mathcal{O}^T$ be the ring of invariants. By geometric invariant theory, the reduced Ptolemy variety is a variety whose coordinate ring is isomorphic to $\mathcal{O}^T$.

For $i=0,1$, let C_i denote the free abelian group generated by the *unoriented* i-cells of $\widehat{M}$, and consider the maps (first studied by Neumann [11])

$$\alpha\colon C_0\to C_1,\qquad \alpha^*\colon C_1\to C_0, \tag{4.2}$$

where α takes a 0-cell to a sum of its incident 1-cells, and α^* takes a 1-cell to the sum of its end points. The maps α and α^* are dual under the canonical identifications $C_i \cong C_i^*$. Also, α is injective, and α^* has cokernel of order 2 (see [11].)

The following is an elementary consequence of the definition of the diagonal action.

Lemma 4.3. The diagonal action $P_2^\sigma(\mathcal{T})$, and the induced action on the coordinate ring $\mathcal{O}$ of $P_2^\sigma(\mathcal{T})$ are given, respectively, by

$$(xc)_i = (\prod_{j=1}^{v} x_j^{\alpha_{ij}})c_i, \qquad x(c^w) = \prod_{j=1}^{v} x_j^{\alpha^*(w)_j} c^w, \tag{4.3}$$

where c^w is the monomial $c_1^{w_1} \cdots c_e^{w_e} \in \mathcal{O}$, $w \in \mathbb{Z}^e$. $\square$

Corollary 4.4. Let $w_1, \dots, w_{e-v}$ be a basis for $\operatorname{Ker}(\alpha^*)$. The monomials $c^{w_1}, \dots, c^{w_{e-v}}$ generate $\mathcal{O}^T$. $\square$

4.1.1. *Examples.* Suppose the 1-skeleton of $\widehat{M}$ looks like in Figure 8 (this is in fact the 1-skeleton of the census triangulation of the Whitehead link complement.) We have

$$\alpha^* = \begin{pmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 2 \end{pmatrix} \tag{4.4}$$

and the action of (x_1, x_2) on a Ptolemy assignment c is given in Figure 9. The kernel of α^*

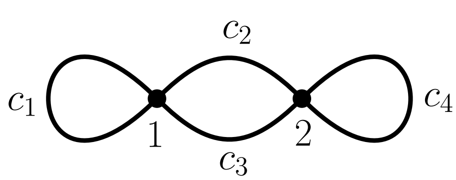

Figure 8. Ptolemy assignment.

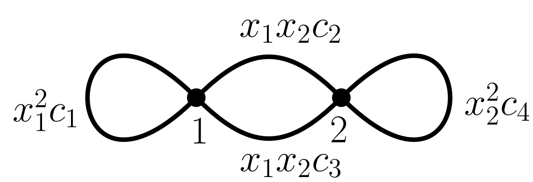

Figure 9. The diagonal action of (x_1, x_2).

is generated by $(0, -2, 0, 1)^t$ and $(-1, 1, 1, 0)^t$, so we have

$$\mathcal{O}^T = \langle c_2^{-2}c_4, c_1^{-1}c_2c_3 \rangle. \tag{4.5}$$

Also note that in each of the examples in Section 3, $x \in \mathcal{O}^T$.

For computations we need a more explicit description of the reduced Ptolemy variety.

Definition 4.5. Let $T\colon \mathbb{Z}^n \to \mathbb{Z}^m$ be a homomorphism. We say that T is *basic* if there exists a subset J of $\{e_1, \dots, e_n\}$ such that T maps $\operatorname{Span}(J)$ isomorphically onto the image of T. Elements of such a set J are called *basic generators* for T.

We identify C_1 and C_0 with $\mathbb{Z}^e$ and $\mathbb{Z}^v$, respectively.

Proposition 4.6. The map $\alpha^*\colon C_1 \to C_0$ is basic. $\square$

The proof will be relegated to Section 4.3, where we shall also give explicit basic generators.

Proposition 4.7. Let $\varepsilon_{i_1}, \dots, \varepsilon_{i_v}$ be basic generators for α^*. The ring of invariants $\mathcal{O}^T$ is isomorphic to $\mathbb{C}[c_1, \dots, c_e]$ modulo the Ptolemy relations and the relations $c_{i_1} = \cdots = c_{i_v} = 1$, i.e. the reduced Ptolemy variety is isomorphic to the subset of $P_2^\sigma(\mathcal{T})$ where the Ptolemy coordinates of the basic generators are 1.

Proof. Let $w_1, \dots, w_{e-v}$ be a basis for $\mathrm{Ker}(\alpha^*)$. Hence, $w_1, \dots, w_{e-v}$ and $\varepsilon_{i_1}, \dots, \varepsilon_{i_v}$ generate C_1. We can thus uniquely express each c_i as a monomial in the w_j's and the c_{i_j}'s. The result now follows from Corollary 4.4. □

Remark 4.8. This is how the Ptolemy varieties are computed in SnapPy.

4.2. **Shapes and gluing equations.** One can assign to each simplex a *shape*

$$z = \sigma_3\sigma_2 \frac{c_{03}c_{12}}{c_{02}c_{13}} \in \mathbb{C} \setminus \{0, 1\}, \tag{4.6}$$

and one can show (see [7, 5]) that these satisfy Thurston's gluing equations. For the geometric representation of a cusped hyperbolic manifold, the shape field (field generated by the shapes) is equal to the invariant trace field, which is in general smaller than the trace field, see Maclachlan–Reid [10].

Remark 4.9. Note that the shapes are elements in $\mathcal{O}^T$.

4.3. **Proof that α^* is basic.** Since α^* has cokernel of order 2, it is enough to prove that there is a set of columns of α^* forming a matrix with determinant ± 2. Recall that the columns of α^* corresponds to 1-cells of $\mathcal{T}$. We shall thus consider graphs in the 1-skeleton of $\widehat{M}$. We recall some basic results from graph theory. All graphs are assumed to be connected.

Definition 4.10. The *incidence matrix* of a graph G with vertices $v_1, \dots, v_k$ and edges $\varepsilon_1, \dots, \varepsilon_l$ is the $k \times l$ matrix I_G whose ijth entry is 1 if v_i is incident to ε_j, and 0 otherwise.

Lemma 4.11. The rank of I_G is $k-1$. If G is a tree, I_G is a $k \times (k-1)$ matrix, and removing any row gives a matrix with determinant ± 1. □

4.3.1. *Case* 1*: A single cusp.* In this case the result is trivial. The matrix representation for α^* is $\begin{pmatrix} 2 & \cdots & 2 \end{pmatrix}$.

4.3.2. *Case* 2*: Multiple cusps, self edges.* Suppose $\widehat{M}$ has a self edge ε_1 (an edge joining a cusp to itself), and consider the graph G consisting of the union of ε_1 with a maximal tree T (see Figure 10). The columns of α^* corresponding to the edges of G then form the matrix

$$B = \left(\begin{array}{c|c} 2 & \\ \hline 0 & I_T \end{array}\right) \tag{4.7}$$

which, by Lemma 4.11, has determinant ± 2.

4.3.3. *Case* 3*: Multiple cusps, no self edges.* Pick a face with edges $\varepsilon_1, \varepsilon_2, \varepsilon_3$, and add edges to form a graph G such that $G \setminus \varepsilon_1$ is a maximal tree (see Figure 11). The corresponding columns form the matrix

$$C = I_G = \left(\begin{array}{c|c} 1 & \\ 0 & I_T \\ 1 & \\ \hline 0 & \end{array}\right) \tag{4.8}$$

which, by Lemma 4.11, has determinant ± 2. This concludes the proof that α^* is basic.

Note that

$$\det(B) = \det\begin{pmatrix} 2 & 1 \\ & 1 \end{pmatrix} = 2, \qquad \det(C) = \det\begin{pmatrix} 1 & 1 & \\ 0 & 1 & 1 \\ 1 & & 1 \end{pmatrix} = 2, \tag{4.9}$$

i.e. only the edges and vertices shown in Figures 10 and 11 contribute to the determinant.

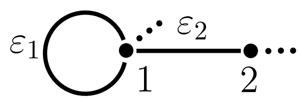

Figure 10. Tree G with 1-cycle; $G \setminus \varepsilon_1$ is a maximal tree.

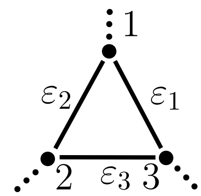

Figure 11. Tree G with 3-cycle; $G \setminus \varepsilon_1$ is a maximal tree.

Remark 4.12. Trees with 1- or 3-cycles are also used in [6, Sec.4.6] to study index structures.

5. The Ptolemy field and the trace field

5.1. **Explicit description of the Ptolemy field.** By Proposition 4.7 any $c \in P_2^\sigma(\mathcal{T})$ is equivalent to a Ptolemy assignment c' whose coordinates for a set of basic generators $\varepsilon_{i_1}, \dots, \varepsilon_{i_v}$ is 1. In particular, it follows that the Ptolemy field (see Definition 1.12) of $c \in P_2^\sigma(\mathcal{T})$ is given by

$$k_c = k_{c'} = \mathbb{Q}\big(\{c_{\varepsilon_1}, \dots, c_{\varepsilon_e}\}\big). \tag{5.1}$$

Definition 5.1. Let $\rho : \pi_1(M) \to \mathrm{PSL}(2, \mathbb{C})$ be a representation. The *trace field* of ρ is the field generated by the traces of elements in the image. We denote it k_ρ.

Our main result is the following. We defer the proof to Section 5.4.

Theorem 5.2. *Let $c \in P_2^\sigma(\mathcal{T})_{\mathrm{red}}$. If the corresponding generic representation ρ of $\pi_1(M)$ in $\mathrm{PSL}(2, \mathbb{C})$ is boundary-non-trivial, the Ptolemy field of c equals the trace field of ρ.* $\square$

Remark 5.3. Note that if $c \in P_2^\sigma(\mathcal{T})_{\mathrm{red}}$ is in a degree 0 component, the Ptolemy field is a number field.

5.2. **The setup of the proof.** Since the natural cocycle is given in terms of the Ptolemy coordinates, it follows that ρ is defined over the Ptolemy field. Hence, the trace field is a subfield of the Ptolemy field.

Fix a maximal tree G with 1 or 3-cycle as in Figures 10 or 11. As explained in Section 4.3 the edges of G are basic generators of α^*. We may thus assume without loss of generality that the Ptolemy coordinates c_i of the edges ε_i of G are 1. By (5.1), it is thus enough to show that the Ptolemy coordinates of the remaining 1-cells are in the trace field.

Let γ denote the (lifted) natural cocycle of c. Then γ assigns to each edge path p in $\widehat{M}$ a matrix $\gamma(p) \in \mathrm{SL}(2, \mathbb{C})$. Let

$$\alpha(a) = \begin{pmatrix} 0 & -a^{-1} \\ a & 0 \end{pmatrix}, \qquad \beta(b) = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}. \tag{5.2}$$

As shown in Figure 3, γ takes long and short edges to elements of the form $\alpha(a)$ and $\beta(b)$, respectively, where a and b are given in terms of the Ptolemy coordinates.

Since ρ is boundary-non trivial, there exists for each cusp i of M a peripheral loop M_i with $\gamma(M_i) \in P$ non-trivial. We shall here refer to such loops as *non-trivial*. Fix such non-trivial loops M_i, once and for all, and let $m_i \neq 0$ be such that $\gamma(M_i) = \beta(m_i)$. For any edge path p with end point on a cusp i we can alter M_i by a conjugation if necessary (this does not change m_i) so that p is composable with M_i.

5.3. **Proof for one cusp.** We first prove Theorem 5.2 in the case where there is only one cusp. In this case, all edges are self edges, and T consists of a single edge ε_1.

Lemma 5.4. For any self edge ε, we have $m_1 c_\varepsilon \in k_\rho$.

Proof. Let X_1 be a peripheral path such that $X_1\varepsilon$ is a loop (see Figure 12), and let x_1 be such that $\gamma(X_1) = \beta(x_1)$. We have

$$\mathrm{Tr}(\gamma(X_1\varepsilon)) = \mathrm{Tr}(\beta(x_1)\alpha(c_\varepsilon)) = \mathrm{Tr}\left(\begin{pmatrix}1 & x_1\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & -c_\varepsilon^{-1}\\ c_\varepsilon & 0\end{pmatrix}\right) = x_1 c_\varepsilon \in k_\rho. \tag{5.3}$$

Applying the same computation to the loop $X_1 M_1 \varepsilon$ yields

$$\mathrm{Tr}(\beta(x_1)\beta(m_1)\alpha(c_\varepsilon)) = (x_1 + m_1)c_\varepsilon \in k_\rho, \tag{5.4}$$

and the result follows. □

Since the Ptolemy coordinate of ε_1 is 1, it follows that $m_1 \in k_\rho$. Since all edges are self edges, we have $c_\varepsilon \in k_\rho$ for all 1-cells ε. This concludes the proof in the one cusped case.

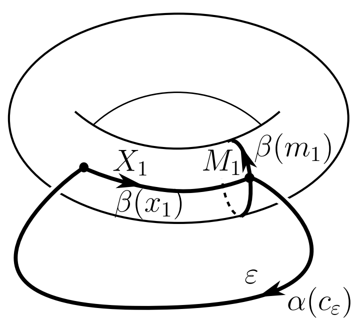

Figure 12. Self edge.

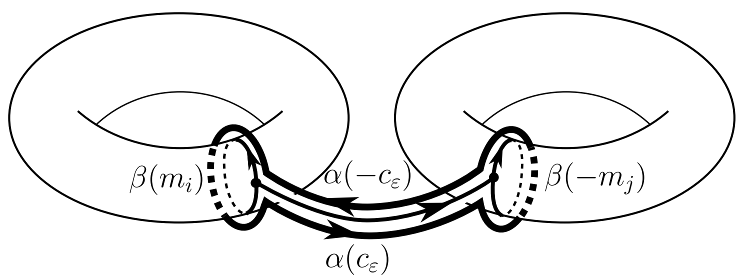

Figure 13. Edge between cusps.

5.4. **The general case.** The general case follows the same strategy, but is more complicated since it involves edge paths between multiple cusps.

Lemma 5.5. If ε is a self edge from cusp i to itself, $m_i c_\varepsilon \in k_\rho$

Proof. The proof is identical to that of Lemma 5.4. □

Lemma 5.6. If two (distinct) cusps i and j are joined by an edge ε in G, we have

$$m_i m_j \in k_\rho. \tag{5.5}$$

Proof. Consider the loop $\varepsilon_j \bar{M}_j \bar{\varepsilon_j} M_i$ shown in Figure 13. A simple computation shows that

$$\mathrm{Tr}(\alpha(c_\varepsilon)\beta(-m_j)\alpha(-c_\varepsilon)\beta(m_i)) = 2 + m_i m_j c_\varepsilon^2. \tag{5.6}$$

Since $\varepsilon \in T$, $c_\varepsilon = 1$, and the result follows. □

More generally, the following holds.

Lemma 5.7. We have $m_i \in k_\rho$ for all cusps i.

Proof. If G is a tree with 1-cycle, then $c_1 = 1$, so Lemma 5.5 implies that $m_1 \in k_\rho$. Inductively applying Lemma 5.6 for the edge ε_j connecting cusp $i = j-1$ and j implies the result. If G is a tree with 3-cycle, the Ptolemy coordinates c_1, c_2 and c_3 are 1, so the edges of the face are labeled by $\alpha(1)$ and $\beta(-1)$ only (see Figure 3). Inserting the peripheral loops M_i as in Figure 14, we obtain

$$\mathrm{Tr}\big(\beta(-1)\beta(m_1)\alpha(1)\beta(-1)\beta(m_2)\alpha(1)\beta(-1)\beta(m_3)\alpha(1)\big) \in k_\rho \tag{5.7}$$

By an elementary computation, the trace equals

$$m_1m_2m_3 - m_1m_2 - m_2m_3 - m_3m_1 + 2 \in k_\rho. \tag{5.8}$$

By Lemma 5.6, $m_i m_j \in k_\rho$, so $m_1 \in k_\rho$. The result now follows as above by inductively applying Lemma 5.6. □

Let ε be an arbitrary 1-cell. If ε is a self edge, Lemmas 5.5 and 5.7 imply that $c_\varepsilon \in k_\rho$. Otherwise, there exists an edge path p in the maximal tree $G \setminus \varepsilon_1$, such that $p * \varepsilon$ is a loop in $\widehat{M}$. By relabeling the cusps and edges if necessary, we may assume that $p = \varepsilon_{i+1} * \varepsilon_{i+2} * \cdots * \varepsilon_j$, where ε_k goes from cusp $k-1$ to cusp k. Pick peripheral paths X_k on cusp k connecting the ends (in M, not $\widehat{M}$) of edges ε_k and ε_{k+1} (see Figure 15). We obtain a loop that can be composed with arbitrary powers of the peripheral loops $M_i, \dots, M_j$. We thus obtain the following traces (where $b_k \in \mathbb{Z}$)

$$\mathrm{Tr}(\beta(x_i + b_i m_i)\alpha(c_{i+1})\beta(x_{i+1} + b_{i+1}m_{i+1})\alpha(c_{i+2})\cdots\beta(x_j + b_j m_j)\alpha(c_\varepsilon)) \in k_\rho. \tag{5.9}$$

It will be convenient to regard $\mathrm{Tr}(\beta(x_i)\alpha(c_{i+1})\beta(x_{i+1})\alpha(c_{i+2})\cdots\beta(x_j)\alpha(c_\varepsilon))$ as a function of variables x_i (disregarding that the x_i's are fixed expressions of the Ptolemy coordinates).

Definition 5.8. Given a function $f(x_1, \dots, x_r)$, let $\Delta_i f$ be the function given by

$$\Delta_i f(h) = f(x_1, \dots, x_i + h, \dots, x_r) - f(x_1, \dots, x_i, \dots, x_r). \tag{5.10}$$

The following is elementary.

Lemma 5.9. If $f(x_1, \dots, x_r)$ is a polynomial where the exponents of all variables x_i are 0 or 1, and where the highest degree term is $a x_1 x_2 \cdots x_r$, we have

$$\Delta_r\big(\cdots\Delta_2\big(\Delta_1 f(h_1)\big)(h_2)\cdots\big) = a h_1 h_2 \dots h_r, \tag{5.11}$$

and is thus independent of the x_i's. □

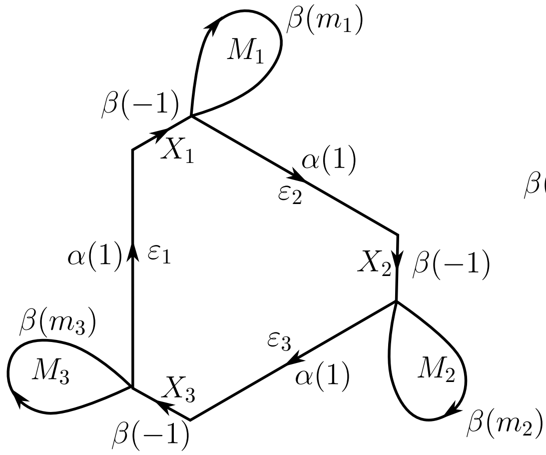

Figure 14. 3-cycle case.

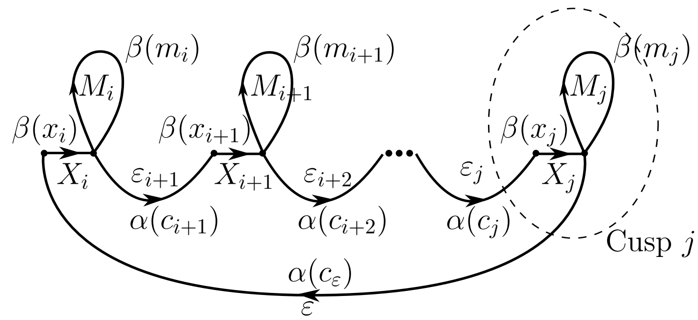

Figure 15. Arbitrary edge ε.

If, e.g., $f(x_1,x_2)=x_1x_2$, we have

(5.12)
$$\Delta_1 f(h_1)=(x_1+h_1)x_2-x_1x_2=h_1x_2,\quad \Delta_2\big(\Delta_1 f(h_1)\big)(h_2)=h_1(x_2+h_2)-h_1x_2=h_1h_2.$$

Lemma 5.10. Let $x_1,\dots,x_r$ be variables and $y_1,\dots,y_r$ be constants. The expression

$$\operatorname{Tr}\big(\beta(x_1)\alpha(y_1)\cdots\beta(x_r)\alpha(y_r)\big) \tag{5.13}$$

is a polynomial in the x_i's whose unique highest degree term is $\prod_{i=1}^r y_i\prod_{i=1}^r x_i$. Moreover, for each monomial term, the exponent of each variable is either 1 or 0.

Proof. This follows by induction on r. □

Applying Lemmas 5.10 and 5.9 to the function

$$f(x_i,\dots,x_j)=\operatorname{Tr}\big(\beta(x_i)\alpha(c_{i+1})\beta(x_{i+1})\alpha(c_{i+2})\cdots\beta(x_j)\alpha(c_\varepsilon)\big), \tag{5.14}$$

we obtain

$$(m_im_{i+1}\cdots m_jc_ic_{i+1}\cdots c_j)c_\varepsilon\in k_\rho. \tag{5.15}$$

Since all m_i's are in k_ρ by Lemma 5.7, and all c_i's are 1 (since $\varepsilon_i\in T$), it follows that c_ε is in k_ρ. This concludes the proof.

5.5. **Proof of Proposition 1.9.** The fact that $\mathcal{R}$ factors, follows from the fact that the diagonal action only changes the decoration (by diagonal elements, c.f. Remark 4.1), not the representation. Since the preimage of the right map in (1.4) is parametrized by choices of lifts, i.e. elements in $Z^1(\widehat{M};\mathbb{Z}/2\mathbb{Z})$, all that remains is to show that the only freedom in the choice of decoration of a boundary-non-trivial representation is the diagonal action. This follows from results in [7]: A decoration is an equivariant map

$$D\colon \widetilde{\widehat{M}}^{(0)}\to \mathrm{PSL}(2,\mathbb{C})/P, \tag{5.16}$$

and is thus determined by its image of lifts $\widetilde{e}_1,\dots,\widetilde{e}_v$ of the cusps of M. The freedom in the choice of $D(\widetilde{e}_i)$ is the choice of a coset gP satisfying that $g\rho(\mathrm{Stab}(\widetilde{e}_i))g^{-1}\subset P$, where

$\mathrm{Stab}(\widetilde{e}_i) \subset \pi_1(M)$ is the stabilizer of $\widetilde{e}_i$, i.e. a peripheral subgroup corresponding to cusp i. Hence, if $\rho(\mathrm{Stab}(\widetilde{e}_i))$ is non-trivial, the freedom is right multiplication by a diagonal matrix (if it is trivial, any coset works). Hence, if ρ is boundary-non-trivial, the only freedom in choosing a decoration is the diagonal action.

6. Ptolemy varieties for $n > 2$

Many of our results generalize in a straightforward way to the higher rank Ptolemy varieties $P_n(\mathcal{T})$. We recall the definition of these below, and refer to [7, 5] for details.

We identify all simplices of $\mathcal{T}$ with a standard simplex

$$\Delta_n^3 = \left\{(x_0, x_1, x_2, x_3) \in \mathbb{R}^4 \;\middle|\; 0 \le x_i \le n,\;\; x_0 + x_1 + x_2 + x_3 = n\right\} \tag{6.1}$$

and regard $\widehat{M}$ as a quotient of a disjoint union $\coprod_{k=1}^s \Delta_{n,k}^3$, with a copy $\Delta_{n,k}^3$ of Δ_n^3 for each simplex k of $\mathcal{T}$. Define $\Delta_n^3(\mathbb{Z}) = \Delta_n^3 \cap \mathbb{Z}^4$ and $\dot{\Delta}_n^3(\mathbb{Z})$ to be $\Delta_n^3(\mathbb{Z})$ with the four vertex points removed. A point in M in the image of $\coprod_{k=1}^s \dot{\Delta}_{n,k}^3(\mathbb{Z})$ is called an *integral point* of M.

6.1. **Definition of the Ptolemy variety.** Assign to each $(t, k) \in \Delta_{n,k}^3(\mathbb{Z})$ a *Ptolemy coordinate* $c_{t,k}$. For each simplex k, we have $|\Delta_{n-2}(\mathbb{Z})| = \binom{n+1}{3}$ *Ptolemy relations*

$$c_{\alpha+1001,k}c_{\alpha+0110,k} + c_{\alpha+1100,k}c_{\alpha+0011,k} = c_{\alpha+1010,k}c_{\alpha+0101,k}, \qquad \alpha \in \Delta_{n-2}(\mathbb{Z}) \tag{6.2}$$

as well as *identification relations*,

$$c_{t,k} = \pm c_{t',k'}, \qquad \text{when} \qquad (t,k) \sim (t',k'). \tag{6.3}$$

Remark 6.1. The signs in 6.3 depend in a non-trivial way on the face pairings (see [5]). For ordered triangulations the signs are always positive. As in Remark 3.3 we can eliminate the identification relations by selecting a representative of each integral point of M.

Definition 6.2. The *Ptolemy variety* $P_n(\mathcal{T})$ is the subset of the affine algebraic set defined by the Ptolemy- and identification relations, consisting of the points where all Ptolemy coordinates are non-zero.

For general n, we denote the group of upper triangular matrices with 1 on the diagonal by N (instead of P). As in 1.2 we have

$$\left\{\begin{matrix}\text{Points in}\\ P_n(\mathcal{T})\end{matrix}\right\} \xleftrightarrow{1-1} \left\{\begin{matrix}\text{Natural } (\mathrm{SL}(n,\mathbb{C}), N)\text{-}\\ \text{cocycles on } M\end{matrix}\right\} \xleftrightarrow{1-1} \left\{\begin{matrix}\text{Generically decorated}\\ (\mathrm{SL}(n,\mathbb{C}), N)\text{-representations}\end{matrix}\right\}. \tag{6.4}$$

6.2. **The diagonal action.** Let D be the group of diagonal matrices in $\mathrm{SL}(n,\mathbb{C})$. We identify D with the torus $(\mathbb{C}^*)^{n-1}$ via the identification

$$(\mathbb{C}^*)^{n-1} \to D, \qquad (a_1, \dots, a_{n-1}) \mapsto \mathrm{diag}(a_1, a_2/a_1, \dots, a_{n-1}/a_{n-2}) \tag{6.5}$$

As in Remark 4.1, we have a diagonal action of the torus $T = D^v$ on the set of decorated representations, where $(D_1, \dots, D_v) \in T$ acts by replacing the coset gN assigned to a vertex w by gD_iN, where i is the cusp corresponding to w. The corresponding action on $P_n(\mathcal{T})$ is described in Lemma 6.4 below.

Let C_1^n be the group generated by the integral points of M, and let $C_0^n = C_0 \otimes \mathbb{Z}^{n-1}$. In Garoufalidis–Zickert [8] we defined maps

$$\alpha\colon C_0^n \to C_1^n, \qquad \alpha^*\colon C_1^n \to C_0^n \tag{6.6}$$

generalizing (4.2). The map α^* takes an integral point (t,k) to $\sum x_i \otimes e_{t_i}$, where x_i is the cusp determined by vertex i of simplex k. We shall not need the definition of α.

Lemma 6.3 (Garoufalidis–Zickert [8])**.** The map α^* is surjective with cokernel $\mathbb{Z}/n\mathbb{Z}$.

By selecting an ordering of the natural generators of C_0^n and C_1^n, we regard α and α^* as matrices. The following is an elementary consequence of (6.4).

Lemma 6.4. The diagonal action of $T = (\mathbb{C}^*)^{v(n-1)}$ on $P_n(\mathcal{T})$ and the corresponding action on the coordinate ring $\mathcal{O}$ of $P_n(\mathcal{T})$ are given, respectively, by

$$(xc)_t = \Big(\prod_{j=1}^{v(n-1)} x_j^{\alpha_{ij}}\Big)c_t, \qquad x(c^w) = \prod_{j=1}^{v(n-1)} x_j^{\alpha^*(w)_j} c^w. \tag{6.7}$$ □

Corollary 6.5. The ring of invariants $\mathcal{O}^T$ is generated by $c^{w_1}, \dots, c^{w_r}$, where $w_1, \dots, w_r$ are a basis for $\operatorname{Ker}(\alpha^*)$, and $r = \operatorname{rank}(C_1^n) - \operatorname{rank}(C_0^n)$.

Definition 6.6. The *Ptolemy field* of a Ptolemy assignment $c \in P_n(\mathcal{T})$ is defined as

$$k_c = \mathbb{Q}(c^{w_1}, \dots, c^{w_r}) \tag{6.8}$$

where $w_1, \dots w_r$ are (integral) generators of $\operatorname{Ker}(\alpha^*)$.

The following is proved in Section 6.4.

Proposition 6.7. The map $\alpha^*\colon C_1^n \to C_0^n$ is basic. □

Corollary 6.8. Let $p_1, \dots, p_{(n-1)v}$ be integral points that are basic generators of C_1^n. The ring $\mathcal{O}^T$ is generated by the Ptolemy relations together with the relations $c_{p_1} = \cdots = c_{p_{(n-1)v}} = 1$. Equivalently, the reduced Ptolemy variety is isomorphic to the subvariety of $P_n(\mathcal{T})$ consisting of Ptolemy assignments with $c_{p_i} = 1$.

Proof. This follows the proof of Proposition 4.7 word by word. □

Remark 6.9. This is how the Ptolemy varieties and Ptolemy fields at [9] are computed.

6.3. **Representations.**

Definition 6.10. Let ρ be an $(\mathrm{SL}(n,\mathbb{C}), N)$-representation, and let I_i denote the image of the peripheral subgroup corresponding to cusp i. We say that ρ is *boundary-non-degenerate* if each I_i has an element whose Jordan canonical form has a single (maximal) Jordan block.

Proposition 6.11. The map

$$\mathcal{R}\colon P_n(\mathcal{T})_{\mathrm{red}} \to \big\{(\mathrm{SL}(n,\mathbb{C}), N)\text{-representations}\big\}\big/\operatorname{Conj}. \tag{6.9}$$

maps onto the generic representations, and the preimage of a generic boundary-non-degenerate representation consists of a single point.

Proof. The proof is identical to the proof in Section 5.5 for $n = 2$. □

Conjecture 6.12. The Ptolemy field of a generic, boundary-non-degenerate representation is equal to its trace field.

Remark 6.13. Much of the theory also works for $\mathrm{PSL}(n,\mathbb{C})$-representations by means of obstruction classes in $H^2(\widehat{M};\mathbb{Z}/n\mathbb{Z})$. When n is even, obstruction classes in $H^2(\widehat{M};\mathbb{Z}/2\mathbb{Z})$ were defined in [7] for representations in $p\mathrm{SL}(n,\mathbb{C}) = \mathrm{SL}(n,\mathbb{C})/\pm I$. For $\mathrm{PSL}(n,\mathbb{C})$-representations, both the Ptolemy field and the trace field are only defined up to nth roots of unity. The generalized obstruction classes are used on the website [9] and will be explained in a forthcoming publication.

6.4. **Proof that α^* is basic.** By Lemma 6.3 we need to prove the existence of integral points such that the corresponding columns of α^* form a matrix of determinant $\pm n$. As in Section 4.3 we split the proof into three cases.

6.4.1. *Basic matrix algebra.* Let I_k be the identity matrix, R_k the sparse matrix whose first row contains entirely of 1's, S_k the sparse matrix whose lower diagonal consists of 1's ($S_1 = 0$), and T_k the sparse matrix whose lower right entry is 1. The index k denotes that the matrices are $k \times k$.

$$R_3 = \begin{pmatrix} 1 & 1 & 1 \\ & & \\ & & \end{pmatrix}, \quad S_3 = \begin{pmatrix} & & \\ 1 & & \\ & 1 & \end{pmatrix}, \quad T_3 = \begin{pmatrix} & & \\ & & \\ & & 1 \end{pmatrix} \tag{6.10}$$

Lemma 6.14. We have

$$\det(I_k + R_k - S_k) = k+1, \qquad \det(I_k + R_k + T_k - S_k) = 2k+1. \tag{6.11}$$

Proof. This follows e.g. by expanding the determinant using the last column. The matrices $I_k + R_k - S_k$ are shown below for $k = 1, 2, 3$ and 4.

$$(2)\,, \qquad \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 2 & 1 & 1 \\ -1 & 1 & \\ & -1 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 2 & 1 & 1 & 1 \\ -1 & 1 & & \\ & -1 & 1 & \\ & & -1 & 1 \end{pmatrix} \tag{6.12}$$

For $I_k + R_k + T_k - S_k$, the only difference is that the lower right entry is now 2. ☐

Lemma 6.15. Let A, B, C, D be $k \times k$, $k \times l$, $l \times k$, and $l \times l$ matrices, respectively, and let $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$. If D is invertible, we have

$$\det(M) = \det(D)\det(A - BD^{-1}C). \tag{6.13}$$ ☐

Proof. This follows from the identity $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) = \left(\begin{smallmatrix} I & B \\ 0 & D \end{smallmatrix}\right)\left(\begin{smallmatrix} A-BD^{-1}C & 0 \\ D^{-1}C & I \end{smallmatrix}\right)$. ☐

6.4.2. *One cusp.* Pick any face of $\mathcal{T}$ and consider the integral points shown in Figures 16 and 17. Let A_n be the $(n-1) \times (n-1)$ matrix formed by the corresponding columns of α^*. The columns are ordered as shown in the figures, and the rows, i.e. the generators $x \otimes e_i$ of C_0^n, are ordered in the natural way (increasing in i). The following is an immediate consequence of the definition of α^*.

Lemma 6.16. The matrix A_n is given by

$$A_{2k+1} = \begin{pmatrix} I_k + R_k + T_k & I_k \\ S_k & I_k \end{pmatrix}, \qquad A_{2k} = \begin{pmatrix} 2 & 0\cdots 01 & 0 \\ 0 & I_{k-1} + R_{k-1} & I_{k-1} \\ 0 & S_{k-1} & I_{k-1} \end{pmatrix} \tag{6.14}$$

Corollary 6.17. The determinant of A_n is $\pm n$.

Proof. This follows from Lemma 6.15 and Lemma 6.14. □

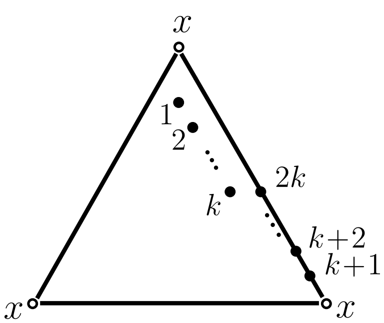

Figure 16. Basic generators, $n = 2k + 1$.

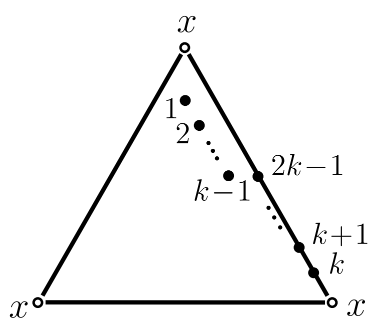

Figure 17. Basic generators, $n = 2k$.

6.4.3. *Multiple cusps, self edges.* Pick a face with a self edge, and extend to a maximal tree with 1-cycle G as in Figure 10. Let $T = G \setminus \varepsilon_1$, and let B_n denote the matrix formed by the columns of α^* corresponding to the face points shown in Figure 18 together with the edge points on T. We order the generators $x_i \otimes e_j$ of C_0^n as follows

$$x_1 \otimes e_1, \dots, x_1 \otimes e_{n-1}, \quad x_2 \otimes e_{n-1}, \dots, x_2 \otimes e_1 \tag{6.15}$$

with a similar scheme for the other vertices. The following is an immediate consequence of the definition of α^*.

Lemma 6.18. The matrix B_n is given by

$$B_n = \left(\begin{array}{c|c} \begin{matrix} I_{n-1} + R_{n-1} \\ S_{n-1} \end{matrix} & \\ \hline 0 & \end{array} \; I_T \otimes \mathbb{Z}^{n-1} \right), \tag{6.16}$$

where $I_T \otimes \mathbb{Z}^{n-1}$ is the matrix obtained from I_T by replacing each non-zero entry by I_{n-1}. □

Corollary 6.19. The determinant of B_n is $\pm n$.

Proof. This follows from

$$\det(B_n) = \pm \det(\begin{pmatrix} I_{n-1} + R_{n-1} & I_{n-1} \\ S_{n-1} & I_{n-1} \end{pmatrix} = \pm n, \tag{6.17}$$

where the second equality follows from Lemmas 6.15 and 6.14. □

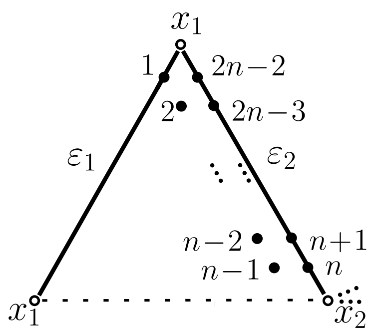

Figure 18. Basic generators, tree with 1-cycle.

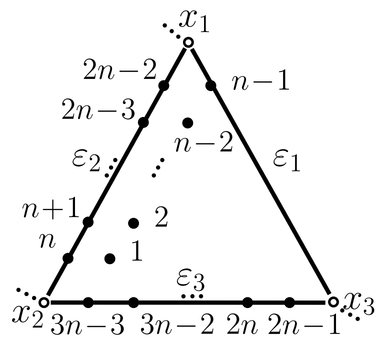

Figure 19. Basic generators, tree with 3-cycle.

6.4.4. *Multiple cusps, no self edge.* Pick a maximal tree with 3-cycle G, and let C_n be the matrix formed by the columns of α^* corresponding to the face points in Figure 19 together with the edge points on $T = G \setminus \varepsilon_1$.

Lemma 6.20. The matrix C_n is given by

$$C_n = \left(\begin{array}{c|c} I_{n-1} & \\ S_{n-1} & I_T \otimes \mathbb{Z}^{n-1} \\ R_{n-1} & \\ \hline 0 & \end{array}\right), \tag{6.18}$$

Corollary 6.21. The determinant of C_n is $\pm n$.

Proof. We have

$$\det(C_n) = \pm\det(M), \qquad M = \begin{pmatrix} I_{n-1} & I_{n-1} & \\ S_{n-1} & I_{n-1} & I_{n-1} \\ R_{n-1} & & I_{n-1} \end{pmatrix}. \tag{6.19}$$

Using Lemma 6.15 with $A = \left(\begin{smallmatrix} I_{n-1} & I_{n-1} \\ S_{n-1} & I_{n-1} \end{smallmatrix}\right)$, $B = \left(\begin{smallmatrix} 0 \\ I_{n-1} \end{smallmatrix}\right)$, $C = \left(\begin{smallmatrix} R_{n-1} & 0 \end{smallmatrix}\right)$, $D = I_{n-1}$, we have

$$\det(M) = \det\begin{pmatrix} I_{n-1} & I_{n-1} \\ S_{n-1} - R_{n-1} & I_{n-1} \end{pmatrix} = \det(I_{n-1} + R_{n-1} - S_{n-1}) = n, \tag{6.20}$$

where the second equation follows from Lemma 6.15, and the third from Lemma 6.14. $\square$

This concludes the proof that α^* is basic.

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